Supergravity Amplitudes, the Double Copy, and Ultraviolet Behavior

超引力振幅、双拷贝与紫外行为

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Abstract

摘要

In this chapter, we present a scattering-amplitudes perspective on supergravity and describe its application to the study of ultraviolet properties of supergravity theories at high-loop orders. The basic on-shell tools that make such calculations feasible are reviewed, including generalized unitarity, color-kinematics duality, and the double-copy construction. We also outline the web of theories connected by the double copy. The results of various calculations of potential ultraviolet divergences are summarized. These include puzzling enhanced ultraviolet cancellations, for which no symmetry-based understanding currently exists, showing that there is much more to learn about the ultraviolet properties of supergravity theories. Finally, we comment on future calculations that should help resolve the puzzles.

在本章中，我们将从散射振幅的角度介绍超引力，并阐述其在研究高圈阶超引力理论紫外性质中的应用。我们综述了让这类计算得以实现的基本在壳工具，包括广义幺正性、色运动学对偶以及双拷贝构造。我们还梳理了通过双拷贝建立联系的理论网络。本文总结了各类潜在紫外发散的计算结果，其中包括目前尚无对称性解释的、令人困惑的增强紫外抵消现象，这表明超引力理论的紫外性质仍有诸多有待探索之处。最后，我们对有助于解决这些谜题的未来计算方向进行了讨论。

Keywords

关键词

Supergravity - Ultraviolet divergences - Unitarity - Color-kinematics duality - Double copy

超引力 - 紫外发散 - 幺正性 - 颜色-运动学对偶 - 二重拷贝

Introduction

引言

Since the discovery of supergravity [1, 2] , fully understanding its hidden symmetries, basic structures, and allowed generalizations has been a highly nontrivial challenge. A surprising hidden structure, shared by many supergravity theories, is the double copy [3-6] which, in its simplest form, implies that gravity scattering amplitudes can be obtained directly from gauge-theory quantities. When combined with the unitarity method [7-9], the double copy has made possible multiloop calculations needed to explicitly determine the ultraviolet properties of various supergravity theories.

自超引力被发现 [1, 2] 以来，全面理解其隐藏对称性、基本结构以及可行的推广形式一直是一项极具挑战性的难题。许多超引力理论共有的一个令人惊讶的隐藏结构就是双拷贝 [3-6]，它最简单的形式意味着引力散射振幅可以直接从规范理论量得到。结合幺正性方法 [7-9]，双拷贝使得明确确定各类超引力理论紫外性质所需的多圈计算成为可能。

Power counting arguments following from the dimensionful nature of Newton's constant suggest that all point-like theories of gravity should be expected to be ultraviolet divergent at some sufficiently high-loop order. Conversely, the absence of such divergences would hint at the existence of some novel structure or symmetry responsible for its cancellation. Therefore, studies of ultraviolet divergences can shed light on basic structures and symmetries present in particular supergravities, exposing properties that are otherwise hidden. Traditional off-shell superspace methods become challenging beyond one loop. The double copy and the unitarity method provided a means to explore multiloop questions in pure supergravities and have been used for such studies at three, four, and five loops [10-17]. In this chapter, we will explain these ideas and emphasize puzzling enhanced ultraviolet cancellations. We will also briefly describe a web of theories linked via the double copy.

由牛顿常数的量纲性质得到的幂次计数论证表明, 所有点状引力理论在足够高的圈阶都会出现紫外发散。反过来说，这类发散的缺失就暗示存在某种新结构或对称性使得发散发生抵消。因此，研究紫外发散可以帮助我们理解特定超引力中的基本结构与对称性，揭示原本隐藏的性质。传统的离壳超空间方法在超过一圈后就会变得难以处理。双拷贝与幺正性方法为探索纯超引力中的多圈问题提供了途径，已经被用于三圈、四圈和五圈的这类研究 [10-17]。本章中，我们将解释这些思想，并重点讨论令人困惑的增强紫外抵消。我们还会简要描述通过双拷贝联系起来的理论网络。

Arguments based on supersymmetry and known duality symmetries reveal that ultraviolet divergences are delayed to surprisingly high-loop orders. In particular, such arguments show that $\mathcal{N} = 8$ supergravity is ultraviolet-finite through at least six loops [10-12, 18-22]. From this perspective, a natural question is then whether all supergravity theories must necessarily diverge in the ultraviolet at some loop order, or whether a hidden structure prevents the appearance of divergences at least for certain special theories.

基于超对称和已知对偶对称性的论证表明，紫外发散会被推迟到非常高的圈阶才出现。具体而言，这类论证表明，至少到六圈，$\mathcal{N} = 8$ 超引力都是紫外有限的 [10-12, 18-22]。从这个角度来看，一个自然的问题是: 是否所有超引力理论都必然会在某一圈阶出现紫外发散，还是说存在隐藏结构，至少对某些特殊理论而言能阻止发散出现?

Studies of unitarity cuts in $D = 4$ suggest the interesting possibility that divergences in $\mathcal{N} = 8$ supergravity may be further delayed [23, 24] in this dimension. So far, the only explicitly known divergence in a pure supergravity arises in $\mathcal{N} = 4$ supergravity at four loops [15]. The interpretation of this divergence is, however, complicated by the presence of a $U(1)$ duality anomaly which might be behind its appearance [25,26]. Such anomalies are absent in $\mathcal{N} \geq 5$ supergravities [27], implying that it is best to focus on these theories. A remarkable case is that of $\mathcal{N} = 5$ supergravity: it does not diverge at four points at the four-loop order [16] despite there being no known symmetry mechanism protecting it [22, 28], and therefore provides the only known four-dimensional example of an enhanced cancellation. It is therefore crucial to determine whether the four-point amplitude in $\mathcal{N} = 5$ supergravity is ultraviolet-finite at five loops. A positive outcome, perhaps supported by the identification of an additional symmetry or structure that can protect against ultraviolet divergences, may point toward the possibility of the all-order finiteness of this theory and ultimately inspire a proof. This would be along the same lines as the proofs of ultraviolet finiteness [29-31] of $\mathcal{N} = 4$ SYM theory [32,33] that followed explicit calculations through three loops [34-37]. Such an all-orders proof in a supergravity theory would be highly nontrivial because of the still mysterious nature of enhanced cancellations. The existence of an undiscovered symmetry or structure would likely have a fundamental impact on our understanding of quantum gravity.

> 对 $D = 4$ 中幺正切割的研究提出了一个有趣的可能性: 该维度下 $\mathcal{N} = 8$ 超引力的发散可能会被进一步推迟 [23, 24] 。到目前为止，纯超引力中唯一被明确发现的发散出现在四圈的 $\mathcal{N} = 4$ 超引力中 [15]。不过，对该发散的解释因存在 $U(1)$ 对偶反常而变得复杂，该反常可能就是发散出现的原因 [25,26]。这类反常在 $\mathcal{N} \geq 5$ 超引力中不存在 [27]，这说明我们最好聚焦于这类理论。一个值得注意的例子是 $\mathcal{N} = 5$ 超引力: 尽管目前没有已知的对称性机制保护它 [22, 28]，它在四圈四顶点处并不发散 [16]，因此这是四维中已知唯一一个增强抵消的例子。所以，明确 $\mathcal{N} = 5$ 超引力的四顶点振幅在五圈是否紫外有限至关重要。如果结果为正——或许会得到额外对称性或结构的支持，它们可以保护理论不出现紫外发散——这可能指向该理论全阶有限的可能性，最终催生相关证明。这与 $\mathcal{N} = 4$ SYM 理论紫外有限性的证明思路一致 [29-31]，这些证明正是在三圈显式计算 [34-37] 后得到的 [32,33]。由于增强抵消的性质至今仍不明确，要在超引力理论中完成这样一个全阶证明会极具难度。若真存在尚未被发现的对称性或结构，它很可能会从根本上影响我们对量子引力的理解。

The study of the ultraviolet properties of theories of gravity has a long history, starting with the seminal work of ’t Hooft and Veltman [38], who showed that pure Einstein gravity is finite at one loop but divergent in the presence of matter, which is also confirmed in other examples [39, 40]. Subsequently, Goroff and Sagnotti showed that pure Einstein gravity diverges at two loops [41-43]. The ultraviolet behavior improves with the addition of supersymmetry. By the late 1970s, it was known that pure ungauged supergravities cannot have divergences prior to three loops [44-46]. The consensus reached from studies in the 1980s was that all pure supergravity theories would likely diverge at the third loop order (see, e.g., Refs. [47-49]), though by making additional assumptions one can raise the loop order of the predicted potential divergences [50].

> 引力理论紫外性质的研究源远流长，始于特霍夫特与韦尔特曼的开创性工作 [38]，他们证明纯爱因斯坦引力在单圈水平是有限的，但在存在物质的情况下发散，这一结论也得到了其他例子的验证 [39, 40]。随后，戈罗夫与萨尼奥蒂证明纯爱因斯坦引力在两圈水平发散 [41-43]。加入超对称后，紫外行为得到改善。截至 20 世纪 70 年代末，人们已经知道纯未定标超引力在三圈之前不会出现发散 [44-46]。从 20 世纪 80 年代的研究中得出的普遍共识是，所有纯超引力理论很可能都会在三圈阶出现发散 (例如参见文献 [47-49])，不过通过额外假设可以提高预测中潜在发散的圈阶 [50]。

The duality between color and kinematics and the associated double copy offer a perspective on gravitational interactions that is radically different from the conventional geometric one. It states that, after a suitable rearrangement of the contributing diagrams, there is a simple procedure to convert gauge-theory scattering amplitudes to gravitational ones via a replacement of color factors by corresponding kinematic factors. It may very well be true that scattering amplitudes in all (super)gravity theories can be written in a double-copy form in terms of gauge theory [51,52]. For the purpose of this review, we will focus on the simplest cases of ungauged $\mathcal{N} \geq 4$ supergravity theories, which have particularly simple double-copy constructions. More generally, the double copy can be applied to non-gauge theories with some Lie algebra symmetry, leading to a veritable web of interrelated theories [6, 53]. Remarkably, the double copy can also be used to relate classical solutions of gravity to those of gauge theory; while no general construction exists, many examples are available - see, for example, Refs. [54,55] and [6] for a review.

色与运动学之间的对偶及相关的双拷贝为引力相互作用提供了一个与传统几何视角完全不同的观点。该理论指出，对贡献图做适当重排后，存在一个简单的步骤: 将色因子替换为对应的运动学因子，即可将规范理论散射振幅转化为引力散射振幅。所有 (超) 引力理论的散射振幅都可以用规范理论写成双拷贝形式，这很可能是正确的 [51,52]。在本综述中，我们将聚焦未定标 $\mathcal{N} \geq 4$ 超引力理论的最简单情形，这类理论拥有格外简单的双拷贝构造。更一般地说，双拷贝可以应用于带有李代数对称性的非规范理论，形成一个名副其实的关联理论网 [6, 53]。值得注意的是，双拷贝还可用于关联引力的经典解与规范理论的经典解；目前虽没有通用构造，但已有大量实例——综述可参见例如文献 [54,55] 和 [6]。

Explicit calculations carried out using these methods have greatly informed and guided our understanding of ultraviolet properties of supergravity theories. For maximally supersymmetric supergravity, which in $D = 4$ is $\mathcal{N} = 8$ super-gravity [56], calculations conclusively demonstrate that the three-loop four-point amplitudes are finite for space-time dimensions $D < 6$ [10, 57] and at four loops for $D < 11/2$ [11]. In $D = 4$, these ultraviolet cancellations were subsequently understood to follow from supersymmetry and the $E_{7(7)}$ duality symmetry of $\mathcal{N} = 8$ supergravity [18,20-22]. A purely supersymmetric explanation was given in Ref. [19] based on the Berkovits' pure spinor formalism [58]. The consensus at the time of this writing is that in $D = 4$ a D^8R^4 counterterm is not forbidden by any of the known symmetries, leading to the expectations of a seven-loop divergence. By analytically continuing to $D = 24/5$, this counterterm corresponds to a five-loop divergence [19]. Such a procedure does not however account for special $D = 4$ properties [23, 24] of these amplitudes of this theory.

利用这些方法开展的显式计算极大地拓展和指导了我们对超引力理论紫外性质的理解。对于最大超对称超引力，即在 $D = 4$ 中的 $\mathcal{N} = 8$ 超引力 [56]，计算最终证明，当时空维度为 $D < 6$ [10, 57] 时，三圈四点振幅是有限的，对于 $D < 11/2$ ，四圈振幅也是有限的 [11]。在 $D = 4$ 中，人们随后认识到这些紫外抵消源于超对称以及 $\mathcal{N} = 8$ 超引力的 $E_{7(7)}$ 对偶对称性 [18,20-22]。基于贝科维茨纯旋子形式体系 [58]，文献 [19] 给出了纯超对称解释。截至本文撰写时，主流共识认为，在 $D = 4$ 中，已知的任何对称性都不禁止 D^8R^4 抵消项，因此人们预期七圈会出现发散。通过解析延拓到 $D = 24/5$ ，该抵消项对应五圈发散 [19]。但这一过程并未说明该理论这些振幅的特殊 $D = 4$ 性质 [23, 24]。

The most interesting cases exhibiting enhanced ultraviolet cancellations, with no currently known symmetry explanation, include:

目前尚无对称性解释的、表现出增强紫外抵消性质的最有趣情形包括:

1. Pure half-maximal supergravity at two loops in $D = 5$ [13, 14, 59, 60]

1. $D = 5$ [13, 14, 59, 60] 中两圈水平的纯半最大超引力

2. Pure $\mathcal{N} = 4$ supergravity at three loops in $D = 4$ [10, 22, 57, 59]

2. $D = 4$ [10, 22, 57, 59] 中三圈水平的纯 $\mathcal{N} = 4$ 超引力

3. $\mathcal{N} = 5$ supergravity at four loops in $D = 4$ [16, 22]

3. $D = 4$ [16, 22] 中四圈水平的 $\mathcal{N} = 5$ 超引力

As already emphasized, the last example is perhaps the most appealing one because the relevant cancellations occur in $D = 4$ and the theory does not suffer from a $U(1)$ anomaly [27] present in $\mathcal{N} = 4$ supergravity [61, 62] . The absence of this anomaly and the known presence of enhanced cancellations make it a candidate ultraviolet-finite theory; verifying this property at the next (five-)loop order is within the reach of currently available methods.

正如前文已经强调的，最后一个例子或许是最吸引人的，因为相关抵消发生在 $D = 4$ 中，且该理论不存在 $\mathcal{N} = 4$ 超引力 [61, 62] 中存在的 $U(1)$ 反常 [27]。这种反常的缺失加上已知存在的增强抵消，让它成为了紫外有限理论的候选者；在下一个 (五) 圈阶验证这一性质，是现有方法可以做到的。

Additional cancellations found in the unitarity cuts of $\mathcal{N} = 8$ supergravity [23, 24] suggest that this theory may very well also display enhanced cancellation at seven loops in $D = 4$, rendering it ultraviolet-finite to at least this loop order. A direct test of this expectation will require further technical advances, given the difficulty of carrying out seven loop calculations.

在 $\mathcal{N} = 8$ 超引力的幺正切割中发现的额外抵消 [23,24] 表明，该理论在 $D = 4$ 的七圈阶也完全可能呈现出增强抵消，使其至少在这个圈阶下仍然是紫外有限的。由于七圈计算难度很大，要直接验证这一预期还需要进一步的技术进步。

What might be the origin of the observed enhanced ultraviolet cancellations? While we do not have a complete picture, in the case of half-maximal supergravity at two loops in $D = 5$, an explanation is revealed by the duality between color and kinematics [13]. Together with the double copy, it relates the two-loop enhanced supergravity cancellations to the cancellation of potential divergences with forbidden color factors in the single-copy gauge theories. This case is particularly simple at two-loop order because the supergravity amplitudes are linear combinations of gauge-theory amplitudes even after the loop integration is carried out. This is connected to the rather simple structure of the four-point two-loop integrand in maximally supersymmetric Yang-Mills theory. A similar analysis at three and higher loops is much more difficult because the simple relation between integrated gauge and gravity amplitudes no longer holds [63]. Other proposals for possible symmetry-based explanations of enhanced cancellation have been put forth in Refs. [64, 65]. New results on generalized symmetries provide new mechanisms which forbid the appearance of certain operators in the effective action [66-68]. It would be important to understand whether generalized symmetries can shed light on the mysterious enhanced cancellation observed in supergravity theories.

观测到的增强紫外抵消可能起源于什么？尽管我们还没有完整的图景，但对于 $D = 5$ 中两圈阶半最大超引力的情况，色与动理学之间的对偶性揭示了一种解释 [13]。它结合双拷贝，将两圈阶超引力的增强抵消和单拷贝规范理论中带禁戒色因子的潜在发散抵消联系起来。这种情况在两圈阶格外简单，因为即使完成圈积分后，超引力振幅仍然是规范理论振幅的线性组合。这和最大超对称杨-米尔斯理论中四点两圈被积函数的简单结构有关。在三圈及更高圈做类似分析要困难得多，因为积分后的规范振幅和引力振幅之间不再满足这种简单关系 [63]。已有文献 [64,65] 基于对称性对增强抵消提出了其他解释。广义对称性的新成果给出了禁止有效行动中出现某些算符的新机制 [66-68]。弄清楚广义对称性能否解释超引力理论中观测到的这种神秘的增强抵消，是十分重要的。

In section ”Tree-Level Amplitudes and Their Properties” of this chapter, we start by first reviewing tree-level amplitudes and the associated on-shell superspaces convenient for describing scattering amplitudes in supersymmetric gauge and gravity theories. This section also explains the duality between color and kinematics and the double copy at tree level. The unitarity method, producing loop-level amplitudes starting from tree amplitudes, and some of its consequences are outlined in section ”Loop-Level Methods”. We also describe the loop-level methods that are used to explore ultraviolet properties that are then summarized in section ”Ultraviolet Properties of Supergravity Theories.” The web of theories, linking various gravitational and nongravitational theories, is summarized in section ”Web of Supergravity Theories.” Finally, in section ”Conclusions and Outlook” we present our conclusions together with an outlook on the application of scattering amplitude methods to supergravity theories.

在本章的“树级振幅及其性质”一节中，我们首先回顾树级振幅，以及便于描述超对称规范理论和引力理论中散射振幅的相应在壳超空间。本节还会解释树级的色动对偶和双拷贝。从树振幅出发得到圈级振幅的幺正方法，及其部分结论，我们会在“圈级方法”一节概述。我们还会介绍用于探究紫外性质的圈级方法，这些性质随后会在“超引力理论的紫外性质”一节总结。连接各类引力理论与非引力理论的理论网络，我们会在“超引力理论网络”一节总结。最后，在“结论与展望”一节中，我们会给出结论，并展望散射振幅方法在超引力理论中的应用。

Tree-Level Amplitudes and Their Properties

树级振幅及其性质

There are two complementary advances that have made it possible to climb multiloop orders in supergravity theories starting from tree-level amplitudes. The first is the realization that the data encoded in tree amplitudes completely constrains loop integrands. In this section we describe tree-level amplitudes which then feed into loop level via on-shell unitarity methods described in section ”Loop-Level Methods.” The second advance follows from the realization that calculations in perturbative gravity do not need to be more complicated than in gauge theory. This is reflected in the double-copy construction that relates gravity amplitudes to Yang-Mills quantities or, in the supersymmetric context, relates supergravity amplitudes to super-Yang-Mills (SYM) quantities. We can encode the spectrum of supergravity theories in terms of the spectrum of two copies of SYM theories. As much of the book-keeping in scattering calculation involves tracking particle states, we will spend the first part of this section describing on-shell superspace in gauge theory, discuss double-copy constructions, and then clarify how this encodes supergravity states.

有两项互补的进展使我们能够从树级振幅出发，推进超引力理论中的多圈阶计算。第一项进展是认识到树振幅编码的数据完全约束了圈被积函数。在本节中，我们将描述树级振幅，这些振幅会通过“圈级方法”一节介绍的在壳幺正性方法应用到圈级。第二项进展源于我们认识到微扰引力的计算不一定比规范理论更复杂。这一点体现在双拷贝构造中: 该构造将引力振幅与杨-米尔斯量联系起来，在超对称语境下则将超引力振幅与超杨-米尔斯 (SYM) 量联系起来。我们可以用两份 SYM 理论的能谱来编码超引力理论的能谱。由于散射计算中的多数整理工作都涉及跟踪粒子态，我们会在本节的前半部分描述规范理论中的在壳超空间，讨论双拷贝构造，随后阐明这种方式如何编码超引力态。

In supersymmetric theories scattering states can be conveniently stratified into so-called on-shell superfields. These superfields realize linearized supersymme-try. In contrast with off-shell Green’s functions which exhibit both the linear and nonlinear part of symmetries, scattering amplitudes require only linearized supersymmetry, with nonlinear terms projected out by the LSZ reduction. With such an organization of the asymptotic states, it is natural to collect amplitudes into superamplitudes which, similar to the on-shell superfields, manifest linearized supersymmetry.

在超对称理论中，散射态可以方便地分层整理为所谓的在壳超场。这些超场实现了线性化超对称。与同时体现对称性线性部分和非线性部分的离壳格林函数不同，散射振幅仅需要线性化超对称，非线性项会被 LSZ 约化投影出去。通过这种渐近态的组织方式，将振幅整理为超振幅是十分自然的——和在壳超场类似，超振幅也彰显了线性化超对称。

Several on-shell superspaces have been put forth; similar to their off-shell counterpart, their formulation is dimension-dependent. Their power however relies on the existence of unconstrained Grassmann variables which form a fundamental representation of the R-symmetry group (Such superspaces are typically referred to as "chiral.") in terms of which superfields are unconstrained polynomials whose coefficients are the physical asymptotic states. Such Grassmann variables were originally introduced in Ref. [69] to yield a supersymmetric extension of twistors and subsequently used in Ref. [70] in the context described here.

目前已经提出了多种在壳超空间；和它们的离壳对应物类似，它们的表述依赖于维度。它们的效力依赖于无约束格拉斯曼变量的存在，这些变量构成了 R 对称群的基本表示 (这类超空间通常被称为“手征”超空间)；以这些变量表示时，超场是无约束多项式，其系数就是物理渐近态。这类格拉斯曼变量最初被引入文献 [69]，用于给出扭量的超对称推广，随后在文献 [70] 中被应用于本文所述的语境。

In this section we discuss the four-dimensional and six-dimensional maximally supersymmetric superspaces and the corresponding tree-level superamplitudes, from which lower-supersymmetric cases can be obtained by truncation, tree-level color-kinematics duality, and the double copy, and some of their consequences.

本节我们将讨论四维和六维的极大超对称在壳超空间，以及对应的树级超振幅——低超对称情形可以通过截断从它们得到——，讨论树级色-运动学对偶、双拷贝，以及它们的部分结论。

Tree-Level Superamplitudes in Various Dimensions

不同维度下的树图级超振幅

In the context of the double copy, it is convenient to encode supergravity states in terms of the states of supersymmetric gauge theories. We begin with maximal supersymmetry. The R symmetry of the maximally supersymmetric gauge theory in four dimensions is $SU(4)$, so the on-shell superspace is constructed in terms of four sets of Grassmann variables η^a , with $a = 1 \cdots 4$ transforming in its fundamental representation. All physical states of this theory are labeled by their helicity and form the vector multiplet of the $\mathcal{N} = 4$ superalgebra. These can be combined into the CPT-self-conjugate on-shell superfield,

在双拷贝的框架下，将超引力态用超对称规范理论的态来编码是十分方便的。我们从最大超对称性情形开始讨论。四维最大超对称规范理论的 R 对称性为 $SU(4)$ ，因此壳上超空间由四组格拉斯曼变量 η^a 构造，其中 $a = 1 \cdots 4$ 在 R 对称性的基础表示下变换。该理论的所有物理态由螺旋度标记，构成 $\mathcal{N} = 4$ 超代数的矢量多重态。这些态可以组合为 CPT 自共轭的壳上超场，

$$\Phi(\eta) = g^+ + \eta^a f_a^+ + \frac{1}{2}\eta^a\eta^b\phi_{ab} + \frac{1}{3!}\varepsilon_{abcd}\eta^a\eta^b\eta^c f^{d-} + \frac{1}{4!}\varepsilon_{abcd}\eta^a\eta^b\eta^c\eta^d g^-.$$

(1)

Similarly, the states of the maximally supersymmetric supergravity theory in four dimensions, $\mathcal{N} = 8$ supergravity, can be organized in a CPT-self-conjugate multiplet which is now built out of eight Grassmann variables, $\eta^A, A = 1 \dots 8$; it exhibits $SU(8)$ R symmetry and contains states up to helicity ±2− as we will see we can encode the $\mathcal{N} = 8$ supergravity states in terms of an outer product of the $\mathcal{N} = 4$ SYM states. Similar to off-shell superspaces, supersymmetry transformations relating the component fields are realized as shifts $\eta^a \mapsto \eta^a + \varepsilon^a$ of the Grassmann variables. The simplest amplitudes are the tree-level so-called maximally helicity violating (MHV) n -point color-ordered amplitudes. The MHV amplitudes are the nonvanishing ones with the maximum imbalance between positive and negative helicity states. As standard for gauge-theory tree-level scattering amplitudes, we also use a color-ordered format which effectively strips the color factors from the amplitudes. (See Refs. [71, 72] for details on color ordering.) With the above organization of asymptotic states, the tree-level MHV n -point color-ordered amplitudes of maximally supersymmetric gauge theory can be packaged into the n -point color-ordered superamplitude [73],

类似地，四维最大超对称超引力理论即 $\mathcal{N} = 8$ 超引力的态可以整理为一个 CPT 自共轭多重态，该多重态由八个格拉斯曼变量 $\eta^A, A = 1 \dots 8$ 构造；它具有 $SU(8)$ R 对称性，包含最高螺旋度为 ±2− 的态。我们将会看到，我们可以将 $\mathcal{N} = 8$ 超引力态编码为 $\mathcal{N} = 4$ 超对称杨-米尔斯 (SYM) 态的外积。和离壳超空间类似，联系分量场的超对称变换通过格拉斯曼变量的平移 $\eta^a \mapsto \eta^a + \varepsilon^a$ 实现。最简单的振幅是树图级的所谓极大螺旋度破坏 (MHV) n 点色序振幅。MHV 振幅是正、负螺旋度态之间螺旋度不平衡程度最大的非零振幅。和规范理论树图散射振幅的标准处理一致，我们同样采用色序格式，该格式可以有效地将颜色因子从振幅中剥离出来。(色序的细节参见文献 [71, 72]。) 通过上述对渐近态的组织，最大超对称规范理论的树图级 MHV n 点色序振幅可以被整理为 n 点色序超振幅 [73],

$$A_n^{\mathrm{MHV}}(1, 2, \cdots, n) = \frac{i}{\prod_{j=1}^{n} \langle j\,(j+1)\rangle} \delta^{(8)}(Q^{a\alpha}), \tag{2}$$

where leg $n+1$ is identified with leg 1, $Q^{a\alpha} \equiv \sum_{j=1}^{n} \lambda_j^\alpha \eta_j^a$ is the total supermomentum, and the delta function imposing its conservation may be thought of as a superpartner of the momentum conservation delta function. In the definition of $Q^{a\alpha}, \lambda_i^\alpha$ and its conjugate $\overline{\lambda}_i^{\dot\alpha}$ are spinors solving the massless mass shell condition for external lines, $p_i^\mu \overline{\sigma}_\mu^{\alpha\dot\alpha} = \lambda_i^\alpha \overline{\lambda}_i^{\dot\alpha}$. The Grassmann delta function can be rewritten as

其中 leg $n+1$ 被识别为 leg $1, Q^{a\alpha} \equiv \sum_{j=1}^{n} \lambda_j^\alpha \eta_j^a$ ，也就是总超动量，而要求超动量守恒的 δ 函数可以被视为动量守恒 δ 函数的超伙伴。在定义中，$Q^{a\alpha}, \lambda_i^\alpha$ 及其共轭 $\overline{\lambda}_i^{\dot\alpha}$ 是满足外线无质量质壳条件 $p_i^\mu \overline{\sigma}_\mu^{\alpha\dot\alpha} = \lambda_i^\alpha \overline{\lambda}_i^{\dot\alpha}$ 的旋量。格拉斯曼 δ 函数可以改写为

$$\delta^{(8)}(Q^{a\alpha}) = \delta^{(8)}\left(\sum_{j=1}^{n} \lambda_j^\alpha \eta_j^a\right) = \prod_{a=1}^{4} \sum_{i<j}^{n} \langle ij\rangle \eta_i^a \eta_j^a. \tag{3}$$

Component amplitude is extracted either by multiplication with the desired states written as superfields and integration over all the Grassmann parameters for all external states [73, 74] or by selecting from Eq.(2) the terms with the desired monomials in the Grassmann parameter; see, e.g., Refs. [74, 75].

分量振幅可以通过两种方式提取: 一种是将其乘上写为超场形式的目标态，再对所有外线的所有格拉斯曼参数积分 [73, 74]；另一种是从式 (2) 中选出含目标格拉斯曼参数单项式的项，例如参见文献 [74, 75]。

We note that the coefficient of the Grassmann delta function in Eq. (2) can be conveniently written in terms of the tree-level MHV color-ordered gluon amplitude, $\mathbb{A}_n^{\text{tree}}(1^-, 2^-, 3^+, \cdots, n^+)$, of non-supersymmetric Yang-Mills theory:

我们注意到，式 (2) 中格拉斯曼 δ 函数的系数可以方便地用非超对称杨-米尔斯理论的树图级 MHV 色序胶子振幅 $\mathbb{A}_n^{\text{tree}}(1^-, 2^-, 3^+, \cdots, n^+)$ 写出:

$$A_n^{\text{MHV}}(1, 2, \cdots, n) = \frac{\mathbb{A}_n^{\text{tree}}(1^-, 2^-, 3^+, \cdots, n^+)}{\langle 12 \rangle^4} \delta^{(8)}(Q^{a\alpha}). \tag{4}$$

The MHV superamplitude of $\mathcal{N} = 8$ supergravity has a similarly close relationship with the tree-level MHV graviton amplitude, $\mathbb{M}_n^{\text{tree}}(1^-, 2^-, 3^+, \cdots, n^+)$, of Einstein's general relativity,

$\mathcal{N} = 8$ 超引力的 MHV 超振幅与爱因斯坦广义相对论的树级 MHV 引力子振幅 $\mathbb{M}_n^{\text{tree}}(1^-, 2^-, 3^+, \cdots, n^+)$ 存在类似的紧密关系，

$$\mathcal{M}_n^{\text{MHV}} = \frac{\mathbb{M}_n^{\text{tree}}(1^-, 2^-, 3^+, \cdots, n^+)}{\langle 12 \rangle^8} \delta^{(16)}(Q^{a\alpha}). \tag{5}$$

On-shell multiplets with reduced supersymmetry can be obtained by suitably truncating the maximally supersymmetric multiplets. One simply sets to zero the fields that do not belong to the desired $\mathcal{N}$ -extended multiplet. In the remaining terms, certain Grassmann variables will, as a result of this truncation, always appear together. They transform under the complement of $SU(\mathcal{N})$ inside the maximal R symmetry. The corresponding MHV superamplitudes are obtained from the maximally supersymmetric MHV amplitude by retaining the maximum number of Grassmann parameters that do not transform under the reduced supersymmetry in the supermomentum-conservation delta function. The simplest example is the amplitudes of pure non-supersymmetric Yang-Mills theory; they are just the pure gluon amplitudes of $\mathcal{N} = 4$ SYM theory, since no superpartners appear in these amplitudes (This projection may be interpreted as the restriction to the invariants under the action of a discrete subgroup of R symmetry and is usually referred to as a "field-theory orbifold." More involved realizations combine the action of a discrete subgroup of R symmetry with the action of the same subgroup of the gauge group. We refer the reader to Ref. [76] a discussion of the effect of such field-theory orbifolds on amplitudes.).

降维超对称的壳外多重态可以通过适当截断极大超对称多重态得到: 只需将不属于目标 $\mathcal{N}$ 扩展多重态的场置零，截断后剩余项中部分格拉斯曼变量必然总是共同出现，这些变量在极大 R 对称性中 $SU(\mathcal{N})$ 的补空间下变换。对于极大超对称 MHV 振幅，保留超动量守恒德尔塔函数中不在约化超对称下变换的最多格拉斯曼参数，即可得到对应的 MHV 超振幅。最简单的例子是纯非超对称杨-米尔斯理论的振幅，它们就是 $\mathcal{N} = 4$ 超对称杨-米尔斯理论的纯胶子振幅，因为这些振幅中不包含超伙伴 (这一投影可以理解为对 R 对称性离散子群作用下不变量的限制，通常被称为“场论轨形”。更复杂的实现会将 R 对称性离散子群的作用与规范群同一子群的作用结合。感兴趣的读者可参考文献 [76] 了解这类场论轨形对振幅的影响。)。

A slightly more involved example is that of the color-ordered MHV tree amplitudes for the minimal gauge multiplets of $\mathcal{N} < 4$ SYM theory. They were shown in Ref. [77] to be given by

一个稍微复杂的例子是 $\mathcal{N} < 4$ 超对称杨-米尔斯理论极小规范多重态的色序树级 MHV 振幅，文献 [77] 已证明它们可由下式给出

$$A_n^{\mathrm{MHV}}(1,2,\ldots,n) = \frac{\prod_{a=1}^{N} \delta^{(2)}(Q^a)}{\prod_{j=1}^{n} \langle j\,(j+1)\rangle} \left(\sum_{i<j}^{n} \langle ij\rangle^{4-N} \prod_{b=N+1}^{4} \eta_i^b \eta_j^b \right). \tag{6}$$

The remaining delta function enforces the conservation of the $\mathcal{N}$ supercharges, while the factor in parenthesis comes from keeping the maximum number of Grassmann parameters that do not transform under the $SU(\mathcal{N})$ R symmetry. See Refs. [77,78] for a more in-depth discussion of $\mathcal{N} < 4$ on-shell superfields and superamplitudes.

剩余德尔塔函数 enforce $\mathcal{N}$ 超荷守恒，括号中的因子来自保留不在 $SU(\mathcal{N})$ R 对称性下变换的最多格拉斯曼参数。关于 $\mathcal{N} < 4$ 壳外超场与超振幅的更深入讨论可参考文献 [77,78]。

With MHV superamplitudes as input, N^mMHV superamplitudes - that is, super-amplitudes which are related by supersymmetry to gluon amplitudes with $m+2$ gluons of negative helicity - can be efficiently constructed either via supersymmetric CSW rules [79] or BCFW recursion relations [74]. While extremely efficient analytically, both approaches yield expressions exhibiting spurious poles, which makes them difficult to use for the construction of loop amplitudes via the unitarity method which we discuss in section ”Loop-Level Methods.” This complication can be avoided by matching them against an ansatz for the amplitude that has only physical poles.

以 MHV 超振幅为输入，可以通过超对称 CSW 规则 [79] 或 BCFW 递推关系 [74] 高效构造 N^mMHV 超振幅——即通过超对称联系到含 $m+2$ 个负螺旋度胶子的胶子振幅的超振幅。尽管两种方法解析计算效率极高，但得到的表达式都包含伪极点，很难用于我们在“圈级方法”一节介绍的幺正性方法构造圈振幅。这一问题可以通过将结果与仅含物理极点的振幅拟设匹配来避免。

Higher-loop calculations require regularization and in section ”Comments on Regularization” we will review some of its important aspects. With dimensional regularization being the preferred method in both non-supersymmetric and component-based supersymmetric calculation and to set the stage, we briefly discuss the $D = 6$ on-shell superspace, constructed in Ref. [80], and the corresponding superamplitudes, which can be used for this purpose.

多圈计算需要正规化，我们将在“关于正规化的评注”一节回顾正规化的若干重要方面。由于维数正规化是非超对称计算和基于分量的超对称计算中最常用的方法，为做好铺垫，我们在此简要讨论文献 [80] 中构造的 $D = 6$ 壳外超空间，以及可用于该目的的对应超振幅。

The six-dimensional massless superspace builds on the six-dimensional spinor-helicity formalism of Ref. [81] (Massive on-shell superspaces can also be constructed; see, e.g., [82] for four dimensions (see also [83-90]) and [91] for five dimensions, but they are outside the scope of this review.). Six-dimensional momenta may be written as 4×4 matrices by contracting them with the six-dimensional chiral Dirac matrices. Then, the on-shell condition demands that this matrix has reduced rank. To obtain the five independent degrees of freedom, they are therefore written as

六维无质量超空间基于文献 [81] 的六维旋量螺旋度形式论构建 (也可以构造有质量壳外超空间；例如四维情况见文献 [82](另见 [83-90])，五维情况见文献 [91]，但这些超出了本综述的讨论范围。)。六维动量可以通过与六维手征狄拉克矩阵缩并写成 4×4 矩阵，壳条件要求该矩阵的秩降阶。为得到五个独立自由度，动量因此可写为

$$p^{AB} = \varepsilon^{ab}\lambda_a^A\lambda_b^B,\ p_{AB} = \frac{1}{2}\varepsilon_{ABCD}p^{CD} = \tilde{\lambda}_A^{\dot{a}}\tilde{\lambda}_B^{\dot{b}}\varepsilon_{\dot{a}\dot{b}}, \tag{7}$$

where the internal $SU(2)$ symmetry acting on the lowercase indices is identified with the massless little-group symmetry. Note that, unlike the four-dimensional case, the spinors λ and $\tilde{\lambda}$ are nontrivially related.

其中作用于小写指标的内部 $SU(2)$ 对称性就是无质量小群对称性。注意和四维情况不同，旋量 λ 与 $\tilde{\lambda}$ 存在非平凡关系。

The choice of Grassmann variables for the construction of superfields relies on two observations: (1) four- and six-dimensional states are related by simple dimensional reduction, and (2) the R symmetry of the four-dimensional theory is not manifestly realized by dimensional reduction, while the six-dimensional R symmetry is manifest. It is then perhaps natural to expect that the six-dimensional $(1,1)$ on-shell superspace should reduce to four-dimensional $\mathcal{N} = 4$ one in which two of the four Grassmann variables are traded for their Fourier conjugates. The corresponding six-dimensional maximally supersymmetric vector superfield is

构造超场时对格拉斯曼变量的选择基于两个观察:(1) 四维与六维态可通过简单维数约化联系起来，(2) 四维理论的 R 对称性无法通过维数约化明显实现，而六维 R 对称性则是明显的。因此我们或许可以自然预期，六维 $(1,1)$ 壳外超空间会约化为四维 $\mathcal{N} = 4$ 超空间，其中四个格拉斯曼变量中的两个会被替换为它们的傅里叶共轭变量。对应的六维最大超对称向量超场为

$$\Phi^{D=6}(\eta,\tilde{\eta}) = \phi + \chi^a\eta_a + \tilde{\chi}_{\dot{a}}\tilde{\eta}^{\dot{a}} + \phi'(\eta)^2 + g^a{}_{\dot{a}}\eta_a\tilde{\eta}^{\dot{a}}$$

$$+\phi''(\tilde{\eta})^2 + \tilde{\lambda}_{\dot{a}}\tilde{\eta}^{\dot{a}}(\eta)^2 + \lambda^a\eta_a(\tilde{\eta})^2 + \phi'''(\eta)^2(\tilde{\eta})^2. \tag{8}$$

Two four-dimensional scalars have been absorbed in the 2×2 matrix $g^a{}_{\dot{a}}$ which contains the physical degrees of freedom of the six-dimensional gluon, leaving four physical scalars, $\phi, \ldots, \phi'''$. With this choice of Grassmann variables, the single-particle supermomenta are

两个四维标量已经被吸收进 2×2 矩阵 $g^a{}_{\dot{a}}$ 中，该矩阵包含六维胶子的物理自由度，剩下四个物理标量 $\phi, \ldots, \phi'''$ 。采用这种格拉斯曼变量的选择后，单粒子超动量为

$$q_i^A = \lambda_{i,a}^A\eta_{+,i}^a,\ \tilde{q}_{iA} = \tilde{\lambda}_{i,A}^{\dot{a}}\tilde{\eta}_{+,i,\dot{a}}. \tag{9}$$

The four-point gauge-theory superamplitude is [80]

四点规范理论超幅为 [80]

$$A_4^{\text{tree}}(1,2,3,4) = \frac{1}{st}\delta^6\left(\sum_{i=1}^4 p_i\right)\delta^4\left(\sum_{i=1}^4 q_i^A\right)\delta^4\left(\sum_{i=1}^4 \tilde{q}_i^A\right). \tag{10}$$

For the three-point and the higher-point superamplitudes, which are substantially more involved, we refer the reader to the original literature [80,92].

对于三点和更高点超幅，其构造要复杂得多，我们建议读者查阅原始文献 [80,92]

As in the four-dimensional superspace, to extract component amplitudes, one multiplies Eq. (10) by the desired superfields (each containing a single nonzero component field) and integrates over all Grassmann variables.

和四维超空间一样，为提取分量振幅，只需将式 (10) 乘上所需的超场 (每个超场仅含一个非零分量场)，然后对所有格拉斯曼变量积分即可

More generally, on-shell superspace provides a convenient way to carry out sums over physical states which are needed when constructing higher-point tree-level superamplitudes either through the super-BCFW on-shell recursion relation [74, 93, 94] or through the super-MHV vertex rules [75, 77, 95-97] and in loop calculations carried out through the generalized unitarity method that will be discussed in section "Loop-Level Methods." The main observation is that integration over Grassmann variables acts as identity operator in the space of states, i.e., in $D = 4$

更一般地，当我们通过超 BCFW 壳递归关系 [74, 93, 94] 或超 MHV 顶点规则 [75, 77, 95-97] 构造高点树级超幅，或是在将在" 圈级方法" 一节讨论的广义幺正性方法进行的圈计算中，都需要对物理态求和，而壳超空间为这类求和提供了一种简便方法。核心结论是，对格拉斯曼变量的积分在态空间上作用为恒等算子，即在 $D = 4$ 中

$$\int d^4\eta \Phi_1(\eta)\Phi_2(\eta) = g_1^+ g_2^- + f_{1,a}^+ f_2^{-a} + \frac{1}{2}\phi_{1ab}\phi_2^{ab} + f_1^{-a} f_{2a}^+ + g_1^- g_2^+, \tag{11}$$

and similarly in $D = 6$ on-shell superspace. The factor of 1/2 accounts for the fact that ϕ_{ab} is antisymmetric.

在 $D = 6$ 壳超空间中也有类似结论。因子 1/2 源于 ϕ_{ab} 是反对称的这一性质

The sum over all the states that can be present on a common external line m of two color-ordered superamplitudes can be written as a Grassmann integral,

两个颜色序化超幅的公共外线 m 上所有可能存在态的求和可以写为一个格拉斯曼积分,

$$\int d^4\eta_m A(1,\ldots,m-1,m)A(m,m+1,\ldots,n). \tag{12}$$

As we will see in later sections, this observation has been used to great effect in $D = 4$ and $D = 6$ to carry out high-order loop calculations by sewing together tree amplitudes using superspace versions of unitarity. Similar Grassmann integrals realize the sum over states in superspace versions of BCFW recursions [74] or MHV vertex rules [96].

我们将在后续章节看到，这一结论已经在 $D=4$ 和 $D=6$ 中发挥了大作用: 通过幺正性的超空间版本，将树振幅拼接起来完成高阶圈计算。在 BCFW 递归的超空间版本 [74] 或 MHV 顶点规则 [96] 中，也用类似的格拉斯曼积分实现态求和

Color-Kinematics Duality and Double Copy

色-动理学对偶与双拷贝

Among the many interesting properties of tree-level gauge-theory scattering amplitudes, color-kinematics duality [4, 5] stands out by providing a bridge between gauge and gravitational theories. In short, it states that in general dimensions, with a suitable reorganization of gauge amplitudes in terms of cubic graphs, the kinematic numerator factors have the same algebraic properties as the corresponding color factors. The existence of such a reorganization is nontrivial. It has been proven at tree level [98, 99] from amplitudes perspective. While one might expect that there should exist a Lagrangian understanding of this property, only partial results are available [100-103]. Analogous constructions of loop-level integrands are available on a case-by-case basis (see, e.g., Refs. [5, 12, 76, 104-124]). The duality between color and kinematics was initially identified [4] as a property of gauge-theory amplitudes and was subsequently extended to gauge theories with matter in various representations of the gauge group [76, 120, 125, 126] and to field theories without gauge fields such as biadjoint ϕ^3 $[127-129]$ and the nonlinear sigma model (NLSM) [53, 130-134]. Perhaps the most remarkable consequence of color-kinematics duality is that it gives, through the double copy, a means to combining amplitudes in distinct theories such that the results are still scattering amplitudes in yet a third field theory.

在树级规范理论散射振幅的诸多有趣性质中，色-动理学对偶 [4, 5] 格外突出，它搭建起了规范理论与引力理论之间的桥梁。简单来说，该性质指出: 在任意维度中，若将规范振幅按三次图做适当重组织，动理学分子因子将具备与对应色因子完全相同的代数性质。这种重组织的存在性是非平凡的，目前已经从振幅视角在树级得到了证明 [98, 99] 。尽管人们认为应当可以从拉格朗日量层面理解这一性质，但目前仅得到了部分结果 [100-103]。圈级被积函数也已有按具体情况逐一构造的类似结果 (例如参见文献 [5, 12, 76, 104-124])。色与动理学之间的对偶最初被发现 [4] 是规范理论振幅的性质，随后被推广到包含属于规范群不同表示物质的规范理论 [76, 120, 125, 126] ，以及不存在规范场的场论，例如双伴随理论 ϕ^3 $[127-129]$ 和非线性 sigma 模型 (NLSM)[53, 130-134]。色-动理学对偶最引人注目的结论或许是，它通过双拷贝提供了一种组合不同理论振幅的方法，组合得到的结果依然是第三个场论的散射振幅。

To illustrate these features, we will focus on gauge theories with matter in adjoint representation of the gauge group. For other theories of vector fields as well as for gauge theories with matter in other representations, we refer the reader to reviews [6, 135] and to the original literature.

为说明这些性质，本文将聚焦于物质处于规范群伴随表示的规范理论。关于矢量场的其他理论，以及物质处于其他表示的规范理论，我们建议读者参阅综述文献 [6, 135] 和原始研究。

Consider the color-dressed tree-level (super)amplitudes of a (supersymmetric) gauge theory, perhaps with matter in the adjoint representation. By suitably multiplying and dividing by propagators, they can be generically written as a sum over diagrams with only three-point vertices:

考虑 (超对称) 规范理论的带色树级 (超) 振幅，该理论可包含伴随表示物质。通过对传播子做适当的乘除操作，这类振幅一般可以写为对所有仅含三点顶点的图的求和:

$$\mathcal{A}_n^{\text{tree}}(1,2,3,\ldots,n)=g^{n-2}\sum_{\text{diags }.i}\frac{n_i c_i}{\prod_{\alpha_i} p_{\alpha_i}^2}. \tag{13}$$

The c_i are color factors obtained by attaching a structure constant f^{abc} to each vertex of the diagram(It is useful to normalize the structure constants with an extra $i\sqrt{2}$ compared to the usual textbook definitions to stay consistent with the conventions used in the amplitudes community.) and δ^{ab} to each internal line, n_i are kinematic numerators which include the Grassmann delta function enforcing the conservation of the supermomentum and possibly further dependence on Grassmann variables, and the $p_{\alpha_i}^2$ in the denominator are the inverse propagators for the various internal lines α_i of the diagram i . There are $(2n-5)$!! such diagrams with n external lines: three at four-point amplitudes, fifteen for five-point amplitudes, etc.

其中 c_i 是色因子，由图的每个顶点连接一个结构常数 f^{abc} 得到 (为了与振幅领域常用的约定一致，相比常规教材定义，给结构常数额外多乘一个 $i\sqrt{2}$ 做归一化会更方便)，给每个内线连接 δ^{ab}；n_i 是动理学分子，包含约束超动量守恒的格拉斯曼德尔塔函数，以及可能对格拉斯曼变量的其他依赖；分母中的 $p_{\alpha_i}^2$ 是图中各条内线 α_i 的逆传播子 i 。对于 n 个外线的过程，共有 $(2n-5)$ 个这类图: 四点振幅有 3 个，五点振幅有 15 个，以此类推。

The two fundamental properties of structure constants are their complete antisymmetry and the Jacobi relation,

结构常数的两个基本性质是全反对称性和雅可比关系，

$$f^{abe}f^{cde}+f^{dae}f^{cbe}+f^{bde}f^{cae}=0, \tag{14}$$

i.e., the commutation relations of the gauge group generators in the adjoint representation. Since the color factors are products of structure constants, the Jacobi relation transfers to one such relation for every internal edge of the diagram. For example, the color factors of three diagrams in Fig. 1 are

即伴随表示中规范群生成元的对易关系。由于色因子是结构常数的乘积，雅可比关系会传递给图的每一条内线对应的色因子。例如，图 1 中三个图的色因子满足

$$c_i\equiv\cdots f^{abe}f^{ecd}\cdots,\ c_j\equiv\cdots f^{dae}f^{ebc}\cdots,\ c_k\equiv\cdots f^{ace}f^{edb}\cdots, \tag{15}$$

where the ellipsis stands for factors common to all three diagrams. Then, Eq. (14) becomes (In any Jacobi relation, the relative signs depend on the chosen relation between labeling of edges and structure constants. Here we chose a clockwise relation.)

其中省略号代表三个图共有的因子，那么式 (14) 可以写为 (在任意雅可比关系中，相对符号取决于我们对边标号和结构常数关系的选择，本文选择顺时针关系)

$$c_i-c_j+c_k=0 \tag{16}$$

or, equivalently, the equality in Fig. 1.

或者等价地，对应图 1 中的等式。

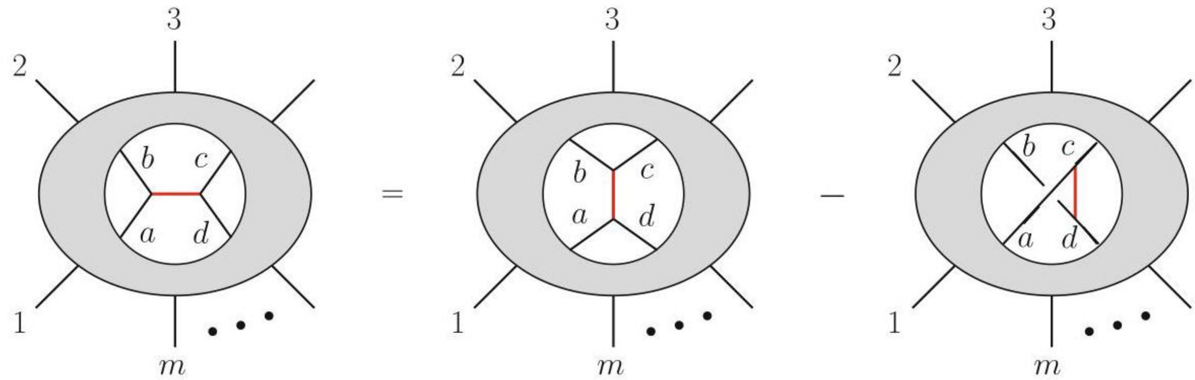


Fig. 1 A Jacobi identity embedded in a generic diagram, either at tree or loop level. Other than the exposed four-point tree subdiagrams, the remaining parts of each of the diagrams is identical

图 1 嵌入任意 (树级或圈级) 图中的雅可比恒等式。除给出的四点树子图外，每个图的其余部分完全相同

As mentioned, the duality between color and kinematics is the statement that kinematic numerators n_i obey algebraic relations in one-to-one correspondence with those obeyed by the color factors c_i . The antisymmetry of the structure constants implies that, under permutations of certain internal legs (e.g., legs a and b in the first diagram of Fig. 1), the color factors are antisymmetric. Thus, imposing this aspect of color-kinematics duality requires that

如前所述，色与动理学的对偶是指: 动理学分子 n_i 满足的代数关系与色因子 c_i 满足的代数关系一一对应。结构常数的反对称性意味着，交换特定内线 (例如图 1 第一个图中的内线 a 和 b) 时，色因子是反对称的。因此，施加色-动理学对偶的这一要求可得

$$c_i \to -c_i \;\Rightarrow\; n_i \to -n_i. \tag{17}$$

Imposing the second aspect of color-kinematics duality, that kinematic numerators obey the dual (kinematic) Jacobi relations, is

色动理学对偶的第二个要求是动理学分子满足对偶 (动理学) 雅可比关系，施加该要求即为

$$c_i = c_j - c_k \;\Rightarrow\; n_i = n_j - n_k. \tag{18}$$

As emphasized in Fig. 1, the kinematic Jacobi identity requires that lines that are common to the three participating graphs are labeled identically.

如图 1 强调的，动理学雅可比恒等式要求三个参与图共有的线必须标注相同。

Apart from exposing structure in the kinematic numerators of color-dressed amplitudes, the color-kinematics duality provides a means to obtain, through double copy, gravitational (super)amplitudes from gauge-theory (super)amplitudes. As we will discuss shortly, the duality between color and kinematics relates the gauge-theory gauge symmetry to the diffeomorphism invariance of gravitational amplitudes.44 For this reason we may relax the definition of the duality to the requirement that kinematic numerators n_i obey algebraic relations in one-to-one correspondence with those obeyed by the color factors c_i and required by gauge invariance. In particular, only the color factor relations that hold for a generic gauge group should be considered, while those valid for special gauge groups or for particular representations should not be imposed on the kinematic numerators.

除了揭示带色振幅的动理学分子结构外，色动理学对偶还提供了一种方法，通过双拷贝从规范理论 (超) 振幅得到引力 (超) 振幅。我们很快就会讨论到，颜色与动理学之间的对偶将规范理论的规范对称性和引力振幅的微分同胚不变性联系起来 44。因此，我们可以放宽对偶的定义，只要求动理学分子 n_i 满足的代数关系与规范不变性要求颜色因子 c_i 满足的代数关系一一对应。特别地，只有对任意规范群都成立的颜色因子关系才需要考虑，而仅对特殊规范群或特定表示成立的关系不需要施加在动理学分子上。

Color-kinematics duality suggests that, in analogy with the construction of color factors from the structure constants of the color algebra, the kinematic numerators may also be constructed from the structure constants of a kinematic algebra. Partial results are available either for special theories or for sectors of generic gauge theories [128, 136-147].

色动理学对偶指出，仿照从颜色代数结构常数构造颜色因子的方法，动理学分子也可以从动理学代数的结构常数构造。目前已有针对特殊理论或一般规范理论部分 Sector 的部分结果 [128, 136-147]。

As formulated above, color-kinematics duality relates amplitudes' building blocks - kinematic numerators - that are not gauge-invariant. They can be changed through generalized gauge transformations, which effectively add arbitrary functions to amplitudes multiplied by combinations of color factors which vanish because of color Jacobi relations. This freedom to modify color-dressed amplitudes has a reflection on color-ordered amplitudes [4]:

按上述表述，色动理学对偶关联的振幅构造模块——动理学分子——并不满足规范不变性。它们可以通过广义规范变换改变: 变换会给振幅添上任意函数，这些函数乘以颜色因子的组合后因颜色雅可比关系消失。这种修改带色振幅的自由度也会反映在色序振幅上 [4]:

$$\sum_{i=2}^{m-1} p_1 \cdot (p_2 + \ldots + p_i) A_m^{\text{tree}} (2, \ldots, i, 1, i+1, \ldots, m) = 0. \tag{19}$$

These relations, usually referred to as the fundamental BCJ amplitude relations, provide a gauge-invariant test for the existence of color-kinematics duality.

这些关系通常被称为基本 BCJ 振幅关系，为色动理学对偶的存在提供了规范不变的检验。

Perhaps the most remarkable aspect of color-kinematics duality is that an amplitude - or, at loop level, the integrand of an amplitude - of a gravitational theory can be constructed by replacing the color factors of one gauge amplitude with the color-dual kinematic numerators of another:

色动理学对偶最引人注一点或许是: 引力理论的振幅 (圈层次上则是振幅的被积函数) 可以通过将一个规范振幅的颜色因子替换为另一个规范振幅满足颜色对偶的动理学分子来构造:

$$c_i \to n_i \tag{20}$$

To be specific and focusing on tree-level amplitudes, given a tree-level n -point gauge-theory amplitude written as in Eq. 13 together with a second n -point amplitude in another gauge theory, possibly with a different field content or gauge group, the supergravity amplitude is

具体来说，我们聚焦树图振幅，给定一个按式 (13) 写出的树图 n 点规范理论振幅，再取另一个规范理论中 (场内容或规范群可以不同) 的第二个 n 点振幅，超引力振幅即为

$$\mathcal{M}_n^{\text{tree}}(1,2,3,\ldots,n) = i\Big(\frac{\kappa}{2}\Big)^{n-2} \sum_{\text{diags}\ .i} \frac{n_i \tilde{n}_i}{\prod_{\alpha_i} p_{\alpha_i}^2}, \tag{21}$$

where the sum runs over the same set of diagrams as in Eq. (13) and the $\tilde{n}_i$ are the kinematic numerators of the second gauge theory. Strikingly the linearized gauge invariance of the Yang-Mills amplitudes has been promoted to linearized diffeomorphism invariance of the gravitational amplitude. Generalizations of this construction to theories with fields in representations other than adjoint have been discussed in Refs. [76, 120, 125, 126, 148-150].

其中求和对与式 (13) 相同的图集合进行，而 $\tilde{n}_i$ 是第二个规范理论的动理学分子。令人惊叹的是，杨-米尔斯振幅的线性化规范不变性被提升为引力振幅的线性化微分同胚不变性。该构造已经被推广到场处于伴随表示以外表示的理论，相关讨论见文献 [76, 120, 125, 126, 148-150]。

In this form, the double-copy relation requires that at least one set of the kinematic numerators, say, $\tilde{n}_i$, is put in a form that manifests color-kinematics duality. Assuming that the n_i also satisfies the duality between color and kinematics, generalized gauge transformations

在这种形式下，双拷贝关系要求至少有一组动理学分子 (例如 $\tilde{n}_i$) 被写为显式满足色动理学对偶的形式。假设 n_i 也满足颜色与动理学的对偶，那么广义规范变换

$$n_i \to n_i' = n_i + \Delta_i, \ \text{satisfying} \ \ 0 = \sum_i \frac{c_i \Delta_i}{\prod_{\alpha_i} p_{\alpha_i}^2}, \tag{22}$$

leave the gauge-theory amplitude unchanged but can yield a representation n_i' that is no longer color-dual. Note that in this case overall contribution to the total amplitude of the generalized gauge transformations Δ_i vanishes as a result of the generic algebraic properties of the color weights because of the algebraic identities they satisfy. Even so, gravitational double-copy construction with at least one manifestly color-dual $\tilde{n}_i$ can proceed and will be equal to double-copy construction using two color-dual representations following:

不改变规范理论振幅，但得到的表示 n_i' 不再满足颜色对偶。注意在这种情况下，广义规范变换 Δ_i 对总振幅的整体贡献因颜色权重满足代数恒等式的一般代数性质而消失。即便如此，只要至少有一个 $\tilde{n}_i$ 是显式颜色对偶的，引力双拷贝构造依然可以进行，且结果等于使用两个颜色对偶表示构造的双拷贝，满足:

$$\sum_i \frac{n'_i \tilde{n}_i}{\prod_{\alpha_i} p^2_{\alpha_i}} = \sum_i \frac{n_i \tilde{n}_i}{\prod_{\alpha_i} p^2_{\alpha_i}} + \sum_i \frac{\Delta_i \tilde{n}_i}{\prod_{\alpha_i} p^2_{\alpha_i}}. \tag{23}$$

The second term on the right vanishes as color-dual $\tilde{n}_i$ satisfy the same algebraic identities as the c_i in Eq. (22).

右侧第二项为零，因为满足颜色对偶的 $\tilde{n}_i$ 与式 (22) 中的 c_i 满足相同的代数恒等式。

A similar argument leads to understanding how linearized gauge invariance is promoted to linearized diffeomorphism invariance. Gauge invariance means that the gauge amplitude will be invariant under shifts of polarizations $\varepsilon_\mu(k) \to \varepsilon_\mu(k) + k_\mu$. With $\delta_i = n_i|_{\varepsilon\to k}$, the amplitude is gauge-invariant only if the shifts satisfy

类似的论证可帮助我们理解线性化规范不变性是如何推广为线性化微分同胚不变性的。规范不变性意味着规范振幅在极化平移 $\varepsilon_\mu(k) \to \varepsilon_\mu(k) + k_\mu$ 下保持不变。在 $\delta_i = n_i|_{\varepsilon\to k}$ 下，振幅仅当平移满足条件时才是规范不变的

$$\sum_i \frac{c_i \delta_i}{\prod_{\alpha_i} p^2_{\alpha_i}} = 0. \tag{24}$$

As the n_i are independent of the color, this again relies only on the algebraic properties of c_i . As a result, if $\tilde{n}_i$ are color-dual, then clearly

由于 n_i 与颜色无关，这再次仅依赖于 c_i 的代数性质。因此，如果 $\tilde{n}_i$ 满足颜色对偶性，显然可得

$$\sum_i \frac{\tilde{n}_i \delta_i}{\prod_{\alpha_i} p^2_{\alpha_i}} = 0. \tag{25}$$

Linearized diffeomorphisms arise from transformations of the type $\varepsilon_{\mu\nu}(k) \to \varepsilon_{\mu\nu}(k) + k_\mu q_\nu + k_\nu q_\mu$, where q is a null reference momenta obeying $k_\mu q^\mu = 0$. With the double-copy polarizations $\varepsilon_{\mu\nu}$ equal to the symmetric traceless part of the product of vector polarizations $\varepsilon_\mu \varepsilon_\nu$ in the double copy, it is straightforward to see that the variation of Eq. (21) under linearized diffeomorphism goes as follows:

线性化微分同胚来源于 $\varepsilon_{\mu\nu}(k) \to \varepsilon_{\mu\nu}(k) + k_\mu q_\nu + k_\nu q_\mu$ 类型的变换，其中 q 是满足 $k_\mu q^\mu = 0$ 的类零参考动量。在双拷贝中，双拷贝极化 $\varepsilon_{\mu\nu}$ 等于矢量极化外积 $\varepsilon_\mu \varepsilon_\nu$ 的对称无迹部分，我们可以直接推导出式 (21) 在线性化微分同胚下的变分形式如下:

$$\mathcal{M}_n^{\text{tree}} \to \mathcal{M}_n^{\text{tree}} - i \left(\sum_i \frac{n_i|_{\varepsilon\to q} \tilde{\delta}_i}{\prod_{\alpha_i} p^2_{\alpha_j}} + \sum_i \frac{\delta_i \tilde{n}_i|_{\varepsilon\to q}}{\prod_{\alpha_i} p^2_{\alpha_i}} \right). \tag{26}$$

Both terms in the second expression vanish under color-dual properties of n_i and $\tilde{n}_i$, leaving the double-copy amplitude invariant under linearized diffeomorphisms.

第二个表达式中的两项都会因 n_i 和 $\tilde{n}_i$ 的颜色对偶性质消失，使得双拷贝振幅在线性化微分同胚下保持不变。

At tree level, generalized gauge transformations can be used to eliminate some of the kinematic numerators. The rest of the color-dual basis numerators can be expressed in terms of color-ordered partial (super)amplitudes. Color-kinematics duality, whether manifest or not, guarantees independence on the choice of kinematic numerators to be eliminated. The net result is the Kawai, Lewellen, and

> 在树图阶，我们可以利用广义规范变换消去部分运动学分子，剩余的颜色对偶基分子可以用颜色序偏 (超) 振幅表示。无论颜色-运动学对偶是否显式，它都保证待消去运动学分子的选择不影响结果，最终得到川井、勒韦伦和

Tye [3] (KLT) relations, derived long ago in the context of string theory. Through five points and written here for superamplitudes, they are

> 戴 [3](KLT) 关系，该关系很早就在弦论框架下推导出来了。此处针对超振幅写出五点及以下的关系，形式如下

$$\mathcal{M}_3^{\text{tree}}(1,2,3) = iA_3^{\text{tree}}(1,2,3)\widetilde{A}_3^{\text{tree}}(1,2,3)\,,$$

$$\mathcal{M}_4^{\text{tree}}(1,2,3,4) = -is_{12}A_4^{\text{tree}}(1,2,3,4)\widetilde{A}_4^{\text{tree}}(1,2,4,3)\,,$$

$$\mathcal{M}_5^{\text{tree}}(1,2,3,4,5) = is_{12}s_{45}A_5^{\text{tree}}(1,2,3,4,5)\widetilde{A}_5^{\text{tree}}(1,3,5,4,2)$$

$$+is_{14}s_{25}A_5^{\text{tree}}(1,4,3,2,5)\widetilde{A}_5^{\text{tree}}(1,3,5,2,4)\ . \tag{27}$$

Starting from two gauge theory color-ordered partial (super) amplitudes A_n^{tree} and $\widetilde{A}_n^{\text{tree}}$,$\mathcal{M}_n^{\text{tree}}$ are tree-level amplitudes in a gravitational theory. Here $s_{ij} = \left(p_i + p_j\right)^2$ are two-particle Mandelstam invariants. Explicit expressions for the KLT relations at n points may be found in Ref. [151] and can be straightforwardly converted to superspace expressions.

> 从两个规范理论的颜色序偏 (超) 振幅 A_n^{tree} 和 $\widetilde{A}_n^{\text{tree}}$,$\mathcal{M}_n^{\text{tree}}$ 出发，即可得到引力理论的树图阶振幅。式中 $s_{ij} = \left(p_i + p_j\right)^2$ 是双粒子曼德尔施塔姆不变量。n 点 KLT 关系的显式表达式可在文献 [151] 中找到，且可以直接转化为超空间表达式。

It is, of course, important to understand what theories arise from double-copy constructions. Given a particular double-copy construction given in terms of kinematic weights of two progenitor gauge theories, the first step toward identifying what theory generates the double-copy amplitudes is to understand the spectrum and the number of manifest supercharges. The latter follows from the observation that the kinematic numerators of each (gauge) theory contain the Grassmann delta function enforcing the conservation of the corresponding supermomentum. Each term in the double-copy amplitude (21) therefore contains the product of the Grassmann delta functions,

> 当然，弄清楚双拷贝构造能得到哪些理论非常重要。对于一个由两个母规范理论运动学权重给出的特定双拷贝构造，确定何种理论产生该双拷贝振幅的第一步，是明确能谱和显式超荷的数量。后者可以通过以下观测得到: 每个 (规范) 理论的运动学分子都包含格拉斯曼 δ 函数，它约束对应超动量守恒。因此双拷贝振幅 (21) 的每一项都包含格拉斯曼 δ 函数的乘积，

$$\delta^{\text{double copy}}\left(\sum \eta\lambda\right) = \delta^{\mathrm{L}}\left(\sum \eta_{\mathrm{R}}\lambda\right)\delta^{\mathrm{R}}\left(\sum \eta_{\mathrm{R}}\lambda\right). \tag{28}$$

The number of supercharges is therefore the sum of the number of supercharges of the two (gauge) theories. For example, by taking the double copying of two theories with 16 supercharges ($\mathcal{N} = 4$ supersymmetry in four dimensions), we obtain a theory with 32 supercharges ($\mathcal{N} = 8$ supersymmetry in four dimensions).

因此总超荷数等于两个 (规范) 理论的超荷数之和。例如，对两个各含 16 个超荷的理论做双拷贝 (四维中 ($\mathcal{N} = 4$ 超对称)，我们得到含 32 个超荷的理论 (四维中 $\mathcal{N} = 8$ 超对称)。

The spectrum of physical states of the double-copy theory is the tensor product of spectra of the two participating gauge theories. To illustrate this, we consider the case that these two theories are both $\mathcal{N} = 4$ SYM theory, whose physical states are organized in the $\mathcal{N} = 4$ on-shell superfield in Eq. (1). Distinguishing the two theories by decorating them with L and R subscripts, the double-copy states are displayed in Table 1. They can be identified as the states of $\mathcal{N} = 8$ supergravity. This is consistent with the previous observation that the double copy of two theories with $\mathcal{N} = 4$ supersymmetry yields a theory with $\mathcal{N} = 8$ supersymmetry.

双拷贝理论的物理态谱是参与的两个规范理论的谱的张量积。为说明这一点，我们考虑两个理论均为 $\mathcal{N} = 4$ 超对称杨-米尔斯理论的情况，该理论的物理态由式 (1) 中的 $\mathcal{N} = 4$ 在壳超场整理得到。通过给两个理论添加 L 和 R 下标加以区分，双拷贝态如表 1 所示。它们可以被识别为 $\mathcal{N} = 8$ 超引力的态。这与之前的结论一致: 两个具有 $\mathcal{N} = 4$ 超对称的理论做双拷贝会得到一个具有 $\mathcal{N} = 8$ 超对称的理论。

The number of supercharges uniquely defines the supergravity theory for $\mathcal{N} \geq 5$ in four dimensions. Together with the spectrum, it continues to do so for $\mathcal{N} = 4$ supergravity theories. Further Lagrangian-based information on three- and perhaps higher-point interactions is necessary to specify the supergravity theory for theories with fewer than 16 supercharges. For example, the $\mathcal{N} = 2$ Maxwell-Einstein theories with a five-dimensional origin are uniquely determined by their spectrum and their three-point interactions [152]. Alternatively, the double-copy amplitudes can be used to construct the Lagrangian of the double-copy theory. (See also section ”Web of Supergravity Theories.”)

四维下，超荷数量唯一确定了 $\mathcal{N} \geq 5$ 超引力理论。对于 $\mathcal{N} = 4$ 超引力理论，结合谱来看，超荷数量依然可以唯一确定该理论。对于超荷数少于 16 的超引力理论，还需要更多基于拉格朗日的三点 (或许还包括更高点) 相互作用信息才能确定该超引力理论。例如，源自五维的 $\mathcal{N} = 2$ 麦克斯韦-爱因斯坦理论就可由其谱和三点相互作用唯一确定 [152]。或者，也可以利用双拷贝振幅来构造双拷贝理论的拉格朗日量。(另见“超引力理论网络”章节。)

Table 1 The states of $\mathcal{N} = 8$ supergravity organized via the double copy. The $SU(8)$ representations are decomposed in representations of the $SU(4) \times SU(4)$ subgroup which is manifest in the construction. The double-copy states can be reorganized into the standard $\mathcal{N} = 8$ multiplet briefly discussed in section ”Tree-Level Superamplitudes in Various Dimensions,” which has manifest $SU(8)$ symmetry and contains 256 physical states

表 1 通过双拷贝整理得到的 $\mathcal{N} = 8$ 超引力的态。在该构造中，$SU(8)$ 表示被明显分解为 $SU(4) \times SU(4)$ 子群的表示。双拷贝态可以重新整理为“不同维度树级超幅”章节中简要讨论的标准 $\mathcal{N} = 8$ 多重态，该多重态具有明显的 $SU(8)$ 对称性，包含 256 个物理态

	$g_R{}^+$	f_R^+	$\phi_{R\tilde{I}\tilde{J}}$	$f_{R\tilde{I}\tilde{J}\tilde{K}}$	$g_{R\tilde{I}\tilde{J}\tilde{K}\tilde{L}}$
$g_L{}^+$	h^+	$\psi_{\tilde{I}}^+$	$A_{\tilde{I}\tilde{J}}^+$	$\chi_{\tilde{I}\tilde{J}\tilde{K}}^+$	$\phi_{\tilde{I}\tilde{J}\tilde{K}\tilde{L}}$
f_{LI}	ψ_I^+	$A_{I\tilde{I}}^+$	$\chi_{I\tilde{I}\tilde{J}}^+$	$\phi_{I\tilde{I}\tilde{J}\tilde{K}}$	$\chi_{I\tilde{I}\tilde{J}\tilde{K}\tilde{L}}$
ϕ_{LIJ}	A_{IJ}^+	$\chi_{IJ\tilde{I}}^+$	$\phi_{IJ\tilde{I}\tilde{J}}$	$\chi_{I\tilde{J}\tilde{I}\tilde{J}\tilde{K}}$	$A_{I\tilde{J}\tilde{I}\tilde{J}\tilde{K}\tilde{L}}^-$
f_{LIJKL}^-	χ_{IJKL}^+	$\phi_{IJKL\tilde{I}}$	$\chi_{IJKL\tilde{I}\tilde{J}}^-$	$A_{IJKL\tilde{I}\tilde{J}\tilde{K}}^T$	$\psi_{IJKL\tilde{I}\tilde{J}\tilde{K}\tilde{L}}^-$
g_{LIJKL}	ϕ_{IJKL}	$\chi_{IJKL\tilde{I}}^-$	$A_{IJKL\tilde{I}\tilde{J}}^-$	$\psi_{IJKL\tilde{I}\tilde{J}\tilde{K}}^-$	$h_{IJKL\tilde{I}\tilde{J}\tilde{K}\tilde{L}}^-$

Having understood the basics of color-kinematics duality and of the double-copy construction, we conclude the section with a discussion of how double copy promotes the global supersymmetry of the gauge theories participating in the double copy to local supersymmetry in the double-copy theory.

在了解了颜色-运动学对偶与双拷贝构造的基础之后，我们在本节最后讨论双拷贝如何将参与双拷贝的规范理论的整体超对称转变为双拷贝理论的局部超对称。

Color-Kinematics Duality and Supersymmetry

色-运动学对偶与超对称

In the case of adjoint fermions in arbitrary dimensions, the duality between color and kinematics can be shown to be equivalent to the existence of supersymmetry, as shown in Ref. [76]. (See also Ref. [153].) The simplest example is Yang-Mills theory minimally coupled to a single adjoint Majorana fermion in D dimensions, described by the Lagrangian

对于任意维度的伴随费米子，正如文献 [76] 所证 (另见文献 [153])，色与运动学之间的对偶等价于超对称的存在。最简单的例子是在 D 维中，杨-米尔斯理论最小耦合单个伴随马约拉纳费米子，由如下拉格朗日量描述

$$\mathcal{L} = \mathrm{Tr}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\overline{\psi}\not{D}^{\mu}\psi\right]. \tag{29}$$

Of course three-point amplitudes are color-dual, and both four-gluon and two-gluon-two-fermion amplitudes automatically respect the duality between color and kinematics. On the other hand, as we will see, requiring the existence of a color-dual form for four-fermion amplitudes leads to a constraint on the space-time dimension.

当然三点振幅满足色对偶，四胶子振幅和双胶子双费米子振幅都自动满足色与运动学之间的对偶。另一方面，正如我们将要看到的，要求四费米子振幅存在色对偶形式会对时空维度给出约束。

Consider the color-dressed amplitude,

考虑带色整体振幅,

$$\mathbb{A}_4^{\text{tree}}\,(1\psi, 2\psi, 3\psi, 4\psi) = i\left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\right), \tag{30}$$

where n_i are kinematic weights given in terms of spinors $\bar{u}_i$ and v_i and the color weights follow the standard Mandelstam channel graphs obeying $c_s + c_t + c_u = 0$. Explicit expressions for the n_i are

其中 n_i 是用旋量 $\bar{u}_i$ 和 v_i 表示的运动学权重，色权重服从满足 $c_s + c_t + c_u = 0$ 的标准曼德尔施塔姆道图。n_i 的显式表达式为

$$2n_s = \left(\bar{u}_1\gamma_\mu v_2\right)\left(\bar{u}_3\gamma^\mu v_4\right),\ 2n_t = \left(\bar{u}_2\gamma_\mu v_3\right)\left(\bar{u}_1\gamma^\mu v_4\right),$$

$$2n_u = \left(\bar{u}_3\gamma_\mu v_1\right)\left(\bar{u}_2\gamma^\mu v_4\right) \tag{31}$$

The spinors satisfy $\bar{u}_i\gamma^\mu v_j = \bar{u}_j\gamma^\mu v_i$ and $\bar{u}_i = v_i^T\mathcal{C}$ for charge conjugation matrix C . In dimensions where a Weyl representation can be chosen, one of the terms above vanishes.

对于电荷共轭矩阵 C ，旋量满足 $\bar{u}_i\gamma^\mu v_j = \bar{u}_j\gamma^\mu v_i$ 和 $\bar{u}_i = v_i^T\mathcal{C}$ 。在可以选择外尔表示的维度中，上述项中有一项为零。

This amplitude is color-dual only if the Dirac matrices are such that

该振幅仅当狄拉克矩阵满足如下条件时才是色对偶的

$$n_s + n_t + n_u = 0 \tag{32}$$

This can only be satisfied in specific dimensions, $D \in \{3, 4, 6, 10\}$. Interestingly, the color-dual requirement in Eq. (32) is exactly the supersymmetric Fierz identity necessary for Eq. (29) to be invariant under supersymmetry transformations. Similar analysis holds for pseudo-Majorana fermions.

这仅在特定维度 $D \in \{3, 4, 6, 10\}$ 中成立。有趣的是，式 (32) 中的色对偶要求恰好是让式 (29) 在超对称变换下不变所必需的超对称费尔斯恒等式。赝马约拉纳费米子也满足类似分析。

Should this relationship between color-dual theories of adjoint fermions and supersymmetry be surprising? Perhaps not from the perspective of double-copy construction of gravitational theories. After all, we can always double-copy adjoint spin-1/2 fermions with vectors to get spin-3/2 fermions, but such amplitudes are only consistent for a theory that can support spin-3/2 particles as part of an associated local supersymmetry multiplet. As we have seen, the maximal dimension for a color-dual gauge theory with adjoint fermions is ten dimensions, therefore bounding the highest dimension for the direct double-copy construction of maximal supergravity as the double copy between two maximally symmetric gauge theories.

伴随费米子色对偶理论与超对称之间的这种关系令人惊讶吗？从引力理论的双拷贝构造的角度来看，或许并不意外。毕竟我们总可以将伴随自旋 1/2 费米子与矢量做双拷贝得到自旋 3/2 费米子，但只有能将自旋 3/2 粒子容纳在相关局域超对称多重态中的理论，这类振幅才自洽。正如我们已经看到的，带伴随费米子的色对偶规范理论的最大维度是十维，因此也限制了最大超引力直接双拷贝构造的最高维度——即两个最大对称规范理论双拷贝得到的最大超引力的维度上限。

Spin-3/2 particles belonging to a local supermultiplet, or gravitini, are an excellent case-study for how constraints of global supersymmetry lead through double copy to constraints of local supersymmetry. The linearized local supersymmetry transformation of a gravitino polarization vector-spinor $u_\mu^\alpha(k)$, analogous to the linearized diffeomorphism transformation of a graviton's polarization, is given:

属于局域超多重态的自旋 3/2 粒子，即引力微子，是研究整体超对称约束如何通过双拷贝导出局域超对称约束的极佳案例。引力微子极化矢量旋量 $u_\mu^\alpha(k)$ 的线性化局域超对称变换，类比于引力子极化的线性化微分同胚变换，表达式如下:

$$u_\mu^\alpha(k) \rightarrow u_\mu^\alpha(k) + k_\mu \xi^\alpha, \tag{33}$$

where ξ_α satisfies the massless Dirac equation $k\xi = 0$. This allows ξ to preserve the γ -tracelessness of the gravitino physical states. Consider how this vector-spinor can arise through double copy - one copy, say, $\tilde{n}_i$, contributes a spinor $\tilde{u}^\alpha(k)$, and the other, n_i , contributes the polarization vector $\varepsilon_i(k)$ associated to a vector field. If color-dual numerators $\tilde{n}_i$ are dependent on a spinor $\tilde{u}$, then they will remain color-dual if $\tilde{u} \rightarrow \xi$ as ξ satisfies all the same properties as $\tilde{u}$. As such the supersymmetry transformation of a double-copy representation (cf. Eq. (26)), given by

其中 ξ_α 满足无质量狄拉克方程 $k\xi = 0$ 。这使得 ξ 可以保持引力微子物理态的 γ 无迹性。我们来讨论这个矢量旋量如何通过双拷贝得到: 一个拷贝，例如 $\tilde{n}_i$ ，贡献旋量 $\tilde{u}^\alpha(k)$ ，另一个拷贝 n_i 贡献与矢量场关联的极化矢量 $\varepsilon_i(k)$ 。如果色对偶分子 $\tilde{n}_i$ 依赖于旋量 $\tilde{u}$ ，那么由于 ξ 满足与 $\tilde{u}$ 完全相同的性质，当 $\tilde{u} \rightarrow \xi$ 时它们仍保持色对偶。因此，双拷贝表示的超对称变换 (参见式 (26)) 为

$$\mathcal{M}_m^{\text{tree}} \rightarrow \mathcal{M}_n^{\text{tree}} + \sum_i \frac{\delta_i \tilde{n}_i|_{\tilde{u}\rightarrow\xi}}{\prod_{\alpha_i} p_{\alpha_i}^2}, \tag{34}$$

is seen to leave the amplitude invariant as the second term vanishes. The amplitudes of the double-copy theory respect linearized local supersymmetry.

由于第二项为零，可证该变换保持振幅不变。双拷贝理论的振幅满足线性化局域超对称。

Loop-Level Methods

圈阶方法

In this section we give an overview of methods that construct loop-level amplitudes from tree-level ones, focusing on those methods that have been used to compute ultraviolet counterterms at higher loops. The primary tools are the unitarity method [7-9, 154] and the double copy [3-6]; a key feature of these methods is that it builds into loop-level amplitudes the simplifications and structures found at tree level. In particular, (super)gravity tree amplitudes constructed via the double copy can be immediately applied to construct loop-level amplitudes, which in turn determine the ultraviolet counterterms. The basic strategy is the same whether we use components or on-shell superspaces [70,77,80,92,97]. (See Refs. [6, 155-157] for reviews and further details.)

在本节中，我们概述由树阶振幅构造圈阶振幅的方法，重点关注已用于计算高圈紫外抵消项的方法。这类方法的核心工具是幺正性方法 [7-9, 154] 与双重拷贝 [3-6]；这些方法的关键特点是，它将树阶发现的简化性质与结构内置到了圈阶振幅中。特别地，通过双重拷贝构造的 (超) 引力树阶振幅可以直接用于构造圈阶振幅，而圈阶振幅反过来又可以确定紫外抵消项。无论我们使用分量形式还是在壳超空间 [70,77,80,92,97]，基本策略都是一致的。(综述与更多细节见文献 [6, 155-157]。)

Overview of the Unitarity Method

幺正方法概述

Unitarity of the S -matrix implies that loop and tree amplitudes are interconnected via discontinuities across branch cuts. Such discontinuities, usually referred to as ”Cutkosky cuts,” are defined as the phase-space integral over products of lower-loop (and perhaps also higher-point) amplitudes. Provided that additional information about the high-energy behavior of amplitudes is available, these discontinuities can be used to reconstruct amplitudes via dispersion integrals.

S 矩阵的幺正性表明，圈振幅与树图振幅通过支割线的不连续性相互关联。这类不连续性通常被称为“卡特斯基割”，定义为对低圈 (也可能是高点) 振幅乘积的相空间积分。只要获得了关于振幅高能行为的额外信息，就可以利用这些不连续性通过色散积分重构振幅。

The modern unitarity method builds on these ideas and on the additional information that (in principle) loop amplitudes can be given in terms of Feynman diagrams, that is, that they are integrals of rational functions with well-understood properties, referred to as ”integrands.” The structure of Feynman diagrams implies that the residues of these rational functions are given in terms of unintegrated products of tree-level amplitudes or of the rational functions corresponding to lower-loop (perhaps higher-point) amplitudes summed over the possible physical states. We refer to these unintegrated products of trees as ”generalized cuts” or simply ”cuts.” For any amplitude these generalized cuts are sufficient to reconstruct its integrand.

现代幺正方法建立在这些思想以及以下额外信息的基础上: 原则上，圈振幅可以用费曼图表示，即它们是性质明确的有理函数的积分，这类有理函数被称为“被积函数”。费曼图的结构表明，这些有理函数的留数由树图振幅未积分乘积，或对应低圈 (可能是高点) 振幅的有理函数对所有可能物理态求和得到。我们将这些树图振幅的未积分乘积称为“广义割”或简称为“割”。对于任意振幅，这些广义割足以重构其被积函数。

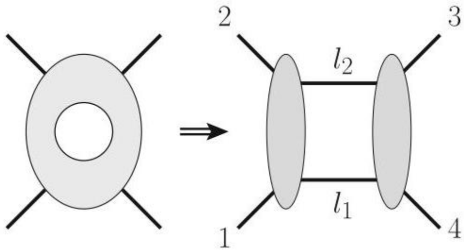
To illustrate the basics of this method, let us discuss the construction of a one-loop amplitude with only massless particles. Such an amplitude is fully constructible from the two-particle (generalized) cuts. One of them is shown in Fig. 2 for the $s = (p_1 + p_2)^2$ cut, and additional inequivalent ones may be obtained by relabeling the external particles. The generalized s -channel cut in Fig. 2 is (This discussion holds for any theory with four-point tree amplitudes. We may equally well choose $\mathcal{N}$ -extended SYM theory, or the NLSM, or some supergravity theory, etc.)

为说明该方法的基础，我们来讨论仅含无质量粒子的单圈振幅构造。这类振幅完全可以通过双粒子(广义)割构造出来。其中一个割如图 2 所示，对应 $s=(p_1+p_2)^2$ 割，其他不等价的割可以通过重新标记外部粒子得到。图 2 所示的广义 s 通道割为 (该讨论适用于任意存在四点树图振幅的理论，我们同样可以选择 $\mathcal{N}$ 扩展超对称杨-米尔斯理论，或是非线性 sigma 模型，或是某些超引力理论等)

$$C_s=\sum_{\text{states}} A^{\text{tree}}(-l_1,1,2,l_3)A^{\text{tree}}(-l_3,3,4,l_1), \tag{35}$$

Fig. 2 The s -channel two-particle cut of the one-loop four-point amplitude. Each blob represents a tree-level amplitude and each exposed line is taken to be on shell

图 2 单圈四点振幅的 s 通道双粒子割。每个 blob 代表一个树图振幅，所有暴露的外线都取在壳条件



where the sum runs over all physical states in the theory. The momenta of all exposed lines, in particular those of internal lines, are placed on shell, $l_i^2=0$, which corresponds to evaluating the residue of the singularity corresponding to those propagators being on shell. The input amplitudes can be either in component form (and the state sum is carried out explicitly) or in superspace form [70, 77, 80, 92, 97] where the state sum is carried out through Grassmann integration, in a similar fashion to Eq. (12) at tree level.

其中求和遍历理论中所有物理态。所有暴露线 (尤其是内线) 的动量都被置于在壳，$l_i^2=0$ ，这对应于对这些传播子在壳对应的奇点计算留数。输入振幅既可以取分量形式 (显式完成态求和)，也可以取超空间形式 [70, 77, 80, 92, 97] ，此时态求和通过格拉斯曼积分完成，方式与树图水平的式 (12) 类似。

Within the context of the unitarity method, there are various strategies for building multiloop amplitudes from tree-level amplitudes. In particular, one may consider cuts recursively where only a single internal line is placed on shell [158- 160], which have been used to construct multiloop planar integrands of $\mathcal{N}=4$ SYM theory [161]. Another strategy, known as prescriptive unitarity [162], has been used to systematically build two-loop amplitudes with an arbitrary number of external legs.

在幺正方法的框架下，存在多种从树图振幅构造多圈振幅的策略。具体而言，可以考虑仅将一条内线置于在壳的递归割方法 [158-160]，该方法已被用于构造 $\mathcal{N}=4$ 超对称杨-米尔斯理论的多圈平面被积函数 [161]。另一种被称为规则幺正性的策略 [162] 已被用于系统构造任意外腿数的双圈振幅。

To construct a complete multiloop integrand, one needs a ”spanning set” of cuts. Such sets have the property that any potential independent contribution is determined by at least one of the generalized cuts. By finding an integrand whose generalized cuts match the appropriate spanning set, one is guaranteed to find the complete integrand. The simplest example of such a spanning set is that for the one-loop four-point massless amplitude, shown in Fig. 2. Any part that is not in this cut (or its relabelings) has a single propagator and thus vanishes in dimensional regularization. A less trivial example is the spanning set of cuts for a massless two-loop four-point amplitude. It is illustrated in Fig. 3, where the complete set is obtained by including all independent relabelings of external legs.

要构造完整的多圈被积函数，需要一组“张成割集”。这类集合满足如下性质: 任何潜在的独立贡献都至少被一个广义割确定。通过找到一个广义割与对应张成集匹配的被积函数，就可以保证得到完整的被积函数。这类张成集最简单的例子就是图 2 所示的单圈四点无质量振幅的集合。任何不在该割 (或其重标记) 中的部分仅含一个传播子，因此在维数正则化中为零。一个非平凡例子是无质量双圈四点振幅的张成割集，如图 3 所示，图中通过包含所有独立的外腿重标记得到了完整集合。

Here we focus on the method of maximal cuts [9], which has been central to computing ultraviolet divergences in supergravity theories, described in the next section. This method uses an overcomplete spanning set of generalized cuts. One begins with the cuts where a maximum number of propagators are placed on shell [9] (the "maximal cuts") and systematically considers all generalized cuts with increasingly fewer cut lines, thus building an integrand that matches all generalized cuts. The resulting integrand is naturally organized in terms of graphs with increasingly higher-multiplicity vertices, referred to as "contact terms." The process terminates when the only remaining potential contact terms exceed power counting constraints of the theory or integrate to zero in dimensional regularization. From this perspective, the building blocks for loop amplitudes are sums of products over m tree amplitudes,

在此我们聚焦最大割方法 [9]，该方法一直是计算超引力理论中紫外发散的核心，我们将在下一节介绍这类计算。该方法使用广义割的过完备张成集。我们从将最多传播子置于在壳的割 [9](即“最大割”) 出发，系统地研究所有切线段数依次更少的广义割，从而构造出匹配所有广义割的被积函数。最终得到的被积函数自然按顶点 multiplicity 依次更高的图组织，这类顶点被称为“接触项”。当剩余所有潜在接触项超出理论的幂次计数约束，或在维度正则化中积分为零时，该过程终止。从这个角度看，圈振幅的构造块是对 m 树振幅乘积的求和，

$$C \equiv \sum_{\text{states}} A^{\text{tree}}_{(1)} A^{\text{tree}}_{(2)} A^{\text{tree}}_{(3)} \cdots A^{\text{tree}}_{(m)}\,, \tag{36}$$

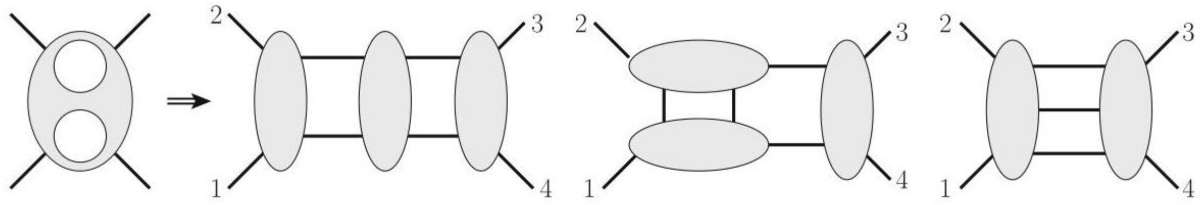


Fig. 3 A sample spanning set of generalized cuts for a massless two-loop four-point amplitude. The complete set is given by the independent relabelings of the four external legs. Each blob represents a tree amplitude and the exposed intermediate lines are on shell

图 3 无质量两圈四点振幅的一组广义割样本张成集。完整集合由四个外腿的独立重标记给出。每个 blob 代表一个树振幅，暴露的中间线为在壳线

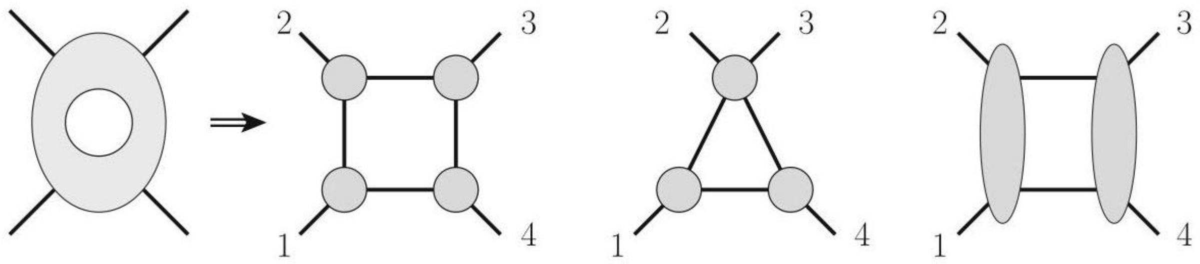


Fig. 4 A sample spanning set of generalized cuts for a massless one-loop four-point amplitude in the method of maximal cuts. The complete set is given by the independent relabelings of the four external legs

图 4 最大割方法中无质量一圈四点振幅的一组广义割样本张成集。完整集合由四个外腿的独立重标记给出

where the sum runs over all physical states that can cross the cuts. As for Eq. (35), the state sum can be carried out either in components or in an on-shell superspace. The amplitudes can either be either gauge or gravity amplitudes. Moreover, in gauge theories they can be either color ordered or color dressed. As a simple example, at one loop the overcomplete spanning set of cuts used in the maximal cut method is shown in Fig. 4.

其中求和遍历穿过割的所有物理态。与式 (35) 一样，态求和可以在分量空间或在壳超空间中完成。对应的振幅可以是规范振幅，也可以是引力振幅。此外，在规范理论中，它们可以是颜色序化的，也可以是带颜色修正的。一个简单的例子可见图 4，它展示了最大割方法中一圈情形下使用的过完备割张成集。

For the case of maximal supersymmetry, there exist a large variety of additional methods and shortcuts for obtaining contributions efficiently, and are related to the relatively simple structure of the amplitudes [9, 161, 163-167]. This makes it possible to carry out explicit calculations, even at five loops.

对于最大超对称的情况，存在大量额外方法与简化技巧可以高效得到贡献，这些方法都与振幅相对简单的结构相关 [9, 161, 163-167]。这使得我们甚至可以完成到五圈的显式计算。

A convenient approach in practical calculations, especially at high-loop orders, is to first construct an ansatz for the diagram numerators containing all possible independent momentum-dependent terms, each with an arbitrary coefficient to be determined by matching the generalized cuts. When using superamplitudes, the numerators will also depend on Grassmann parameters. To simplify the ansatz, it is also convenient to enforce certain auxiliary constraints, such as manifest crossing symmetry, an upper bound on the number of loop momentum factors in each term, and manifest color-kinematics duality. If the ansatz fails to match the generalized cuts, then the ansatz needs to be enlarged by removing auxiliary constraints. This procedure is most straightforward for the four-point amplitudes of $\mathcal{N} = 4$ SYM theory, where it turns out that the ratio between the loop integrand and the tree amplitudes is a rational function of Lorentz-invariant scalar products, at least through five loops and likely beyond this [9, 165, 168-170]. Higher-point loop amplitudes are no longer proportional to the corresponding tree amplitudes [7, 8, 171]; because of this, the numerator ansatz must also contain Lorentz-invariant products of momenta and polarization vectors or tensors, in addition to products of momenta.

在实际计算中，尤其是高圈阶计算，一种简便方法是先对图的分子构造一个包含所有可能独立动量依赖项的拟设，每个项带有一个任意系数，待通过匹配广义割确定。使用超振幅时，分子也会依赖格拉斯曼参数。为简化拟设，引入 certain auxiliary constraints 也很方便，例如显化交叉对称性、给每个项中圈动量因子的数量设定上界，以及显化色-动对偶。如果拟设无法匹配广义割，则需要移除辅助约束来扩大拟设。这个过程对 $\mathcal{N} = 4$ 超对称杨-米尔斯理论的四点振幅最为直接，结果表明至少到五圈 (很可能五圈以上也成立)，圈被积函数与树振幅的比是洛伦兹不变标量积的有理函数 [9, 165, 168-170]。高点圈振幅不再与对应树振幅成正比 [7, 8, 171]；因此，分子拟设除动量乘积外，还必须包含动量与极化矢量或张量的洛伦兹不变乘积。

The generalized unitarity method covers equally well both gauge and gravitational theories. However, it is generally more efficient to construct loop amplitudes in gravitational theories by first constructing corresponding gauge-theory loop integrands that satisfy the duality between color and kinematics and then using the double copy [5]. At five-loop order, where it is difficult to find such duality-satisfying representations, the generalized double copy provides a means to judiciously use the double copy to promote gauge-theory loop integrands to gravitational ones. We will review this in section ”Generalized Double Copy.”

广义幺正性方法对规范理论和引力理论都同样适用。但一般来说，在引力理论中构造圈振幅效率更高的方式是: 先构造满足色-动对偶的对应规范理论圈被积函数，再使用双拷贝 [5]。在五圈阶很难找到满足该对偶的表示，而广义双拷贝提供了一种方法，可以合理利用双拷贝将规范理论圈被积函数转化为引力的被积函数。我们会在“广义双拷贝”一节回顾这部分内容。

Example: The One-Loop Four-Point Integrand of $\mathcal{N} = 4$ SYM Theory

示例: $\mathcal{N} = 4$ SYM 理论的单圈四点被积函数

To illustrate the generalized unitarity method, we discuss a simple example: the on-shell superspace form of the one-loop four-point integrand of $\mathcal{N} = 4$ SYM theory, originally discussed in Refs. [74, 77, 97, 172]. As discussed, the spanning set of generalized supercuts contains a single element, C_s , and its permutations, as shown in Fig. 2. We will interpret it as a color-ordered supercut, so the superamplitudes used to construct it are color-ordered superamplitudes. The sum over states is obtained by carrying out the Grassmann integrals,

为阐明广义幺正性方法，我们讨论一个简单例子: $\mathcal{N} = 4$ SYM 理论单圈四点被积函数的壳上超空间形式，该内容最初在文献 [74, 77, 97, 172] 中讨论。如前文所述，广义超割的生成集仅包含一个元素 C_s 及其排列，如图 2 所示。我们将其解释为颜色序化超割，因此构造它所用的超振幅都是颜色序化超振幅。态求和通过执行格拉斯曼积分得到，

$$C_s = \int d^4\eta_{l_1} \int d^4\eta_{l_3} A_4^{\mathrm{MHV}}(-l_1, 1, 2, l_3)\, A_4^{\mathrm{MHV}}(-l_3, 3, 4, l_1)\,, \tag{37}$$

where the MHV superamplitudes are given by relabeling Eq. (4) for $n = 4$. In simple cases it is straightforward to evaluate the state sums by direct Grassmann integration, but for more complex cuts, it is useful to either use index diagrams that track all internal-state configurations or to systematically solve the system of linear equations implied by the Grassmann delta functions [77]. Here we illustrate the latter method and refer the reader to Ref. [77] for further details on the definition and use of index diagrams.

其中 MHV 超振幅由 $n=4$ 的式 (4) 重新标记得到。在简单情形下，可以通过直接格拉斯曼积分轻松计算态求和，但对于更复杂的割，使用追踪所有内态构型的指标图，或系统求解格拉斯曼 δ 函数蕴含的线性方程组会更方便 [77]。本文我们演示后一种方法，关于指标图的定义和使用的更多细节，请读者参考文献 [77]。

Regularization, usually in the form of dimensional regularization, is needed to properly recover the IR and ultraviolet properties of the theory. To this end, the construction of the integrand must be done in an arbitrary dimension D . It is often convenient however to first construct the integrand in $D=4$ dimensions and subsequently verify its validity in an arbitrary dimension and suitably modify it if necessary. We will return to this point in section "Comments on Regularization"; in this section we will concern ourselves with the four-dimensional integrand of the four-point one-loop amplitude of $\mathcal{N}=4$ SYM theory, which turns out to be the same for all $D<10$.

要正确得到理论的红外和紫外性质，需要正则化，通常采用维数正则化的形式。为此，被积函数必须在任意维度 D 下构造。不过通常更简便的做法是先在 $D=4$ 维构造被积函数，随后验证它在任意维度下的有效性，必要时对其做适当修改。我们会在“正则化注记”一节回到这一点；本节我们讨论 $\mathcal{N}=4$ SYM 理论四点单圈振幅的四维被积函数，结果表明该被积函数对所有 $D<10$ 都是相同的。

The η integration acts on the two supermomentum delta functions contained in the tree-level superamplitudes entering the cut (37),

η 积分作用在出现在割 (37) 中的树-level 超振幅所含的两个超动量 δ 函数上，

$$\delta^{(8)}\left(\lambda_{l_3}^{\alpha}\eta_{l_3}^{a}-\lambda_{l_1}^{\alpha}\eta_{l_1}^{a}+\lambda_{1}^{\alpha}\eta_{1}^{a}+\lambda_{2}^{\alpha}\eta_{2}^{a}\right)\delta^{(8)}\left(\lambda_{l_1}^{\alpha}\eta_{l_1}^{a}-\lambda_{l_3}^{\alpha}\eta_{l_3}^{a}+\lambda_{3}^{\alpha}\eta_{3}^{a}+\lambda_{4}^{\alpha}\eta_{4}^{a}\right). \tag{38}$$

We may interpret the delta functions as imposing a set of linear equations on the η_i^a that we solve. Some of these constraints represent the overall supermomentum conservation and can always be isolated by taking suitable linear combinations of the arguments of the delta functions. In the case at hand, this is realized by simply adding the argument of the first delta function to the argument of the second one. Since this delta function does not depend on the integration variable, it can be taken outside of the Grassmann integral. The cut (37) then becomes

我们可以将 δ 函数理解为对我们要求解的 η_i^a 施加一组线性方程。其中部分约束代表整体超动量守恒，总能通过对 δ 函数的自变量做适当线性组合将其分离出来。在当前情形下，只需将第一个 δ 函数的自变量加到第二个 δ 函数的自变量上即可。由于该 δ 函数不依赖积分变量，可以将它提出格拉斯曼积分。此时割 (37) 变为

$$C_s=-\delta^{(8)}\left(\sum_{i=1}^{4}\lambda_i^{\alpha}\eta_i^{a}\right)\frac{\int d^4\eta_{l_1}d^4\eta_{l_3}\delta^{(8)}\left(\lambda_{l_3}^{\alpha}\eta_{l_3}^{a}-\lambda_{l_1}^{\alpha}\eta_{l_1}^{a}+\lambda_{1}^{\alpha}\eta_{1}^{a}+\lambda_{2}^{\alpha}\eta_{2}^{a}\right)}{\langle l_1 1\rangle\langle 12\rangle\langle 2l_3\rangle\langle l_3 l_1\rangle\langle l_3 3\rangle\langle 34\rangle\langle 4l_1\rangle\langle l_1 l_3\rangle}.$$

(39)

The Grassmann integrals can now be evaluated one by one for each R-symmetry index. Choosing, for example, $a=1$, the fermionic integration is

现在我们可以对每个 R 对称性指标逐一计算格拉斯曼积分。例如，取 $a = 1$ ，费米子积分就是

$$\int d\eta_{l_1}^a d\eta_{l_3}^a \delta^{(2)}\left(\lambda_{l_3}^\alpha \eta_{l_3}^a - \lambda_{l_1}^\alpha \eta_{l_1}^a + \lambda_1^\alpha \eta_1^a + \lambda_2^\alpha \eta_2^a\right) = -\langle l_3 l_1 \rangle, \tag{40}$$

which follows from integrating the form of delta function in Eq. (3). The other three cases $a = 2, 3, 4$, similarly, give the same factor. Thus, the Grassmann integration gives a factor of $\langle l_3 l_1 \rangle^4$.

这可由对式 (3) 的 δ 函数形式积分得到。另外三种情形 $a = 2, 3, 4$ 同理给出相同因子。因此，格拉斯曼积分给出一个因子 $\langle l_3 l_1 \rangle^4$ 。

In more complicated cases, it is more efficient to approach the Grassmann integration in Eq. (39) more systematically, viewing the problem as solving a system of linear constraints,

在更复杂的情形中，将问题视为求解线性约束系统，更系统地处理式 (39) 的格拉斯曼积分效率更高，

$$\lambda_{l_1}^\alpha \eta_{l_1}^a - \lambda_{l_3}^\alpha \eta_{l_3}^a = \lambda_1^\alpha \eta_1^a + \lambda_2^\alpha \eta_2^a, \; a = 1, \dots, 4. \tag{41}$$

There are a total of eight constraints matching the eight integration variables, $\eta_{l_1}^a$ and $\eta_{l_3}^a$, which are therefore completely fixed. The integral is then given by the Jacobian of the matrix of the coefficients of the linear equations,

总共有 8 个约束，对应 8 个积分变量 $\eta_{l_1}^a$ 和 $\eta_{l_3}^a$ ，因此这些变量被完全固定。积分由线性方程系数矩阵的雅可比行列式给出，

$$J = \overset{4}{\det} \begin{vmatrix} \lambda_{l_1}^1 & -\lambda_{l_3}^1 \\ \lambda_{l_1}^2 & -\lambda_{l_3}^2 \end{vmatrix} = \langle l_1 l_3 \rangle^4, \tag{42}$$

which matches the result obtained above by evaluating the Grassmann integrals one by one.

这与我们上文逐一计算格拉斯曼积分得到的结果一致。

Thus, the s-channel cut (37) of the four-point superamplitude is

因此，四点超振幅的 s 道割 (37) 为

$$C_s = -\delta^{(8)}\left(\sum_{i=1}^n \lambda_i^\alpha \eta_i^a\right) \frac{\langle l_1 l_3 \rangle^4}{\langle l_1 1 \rangle \langle 12 \rangle \langle 2 l_3 \rangle \langle l_3 l_1 \rangle \langle l_3 3 \rangle \langle 34 \rangle \langle 4 l_1 \rangle \langle l_1 l_3 \rangle}.$$

(43)

To put this into a form reminiscent of the result obtained from Feynman diagrams, we rationalize the denominators using, for example,

为将其写成类似费曼图结果的形式，我们例如通过下式对分母有理化:

$$\frac{1}{\langle 2l_3\rangle} = -\frac{[2l_3]}{(p-k_1)^2}, \tag{44}$$

where we set $l_1 = p, l_3 = p - k_1 - k_2$ and we used the on-shell conditions $l_1^2 = l_3^2 = 0$. These simplifications lead to

其中我们取 $l_1 = p, l_3 = p - k_1 - k_2$ ，并使用了壳条件 $l_1^2 = l_3^2 = 0$ 。这些化简得到

$$C_s = iA_4^{\mathrm{MHV}} \frac{N_s}{(p-k_1)^4 (p+k_4)^4}, \tag{45}$$

where the numerator $\mathcal{N}$ is given by

其中分子 $\mathcal{N}$ 由下式给出

$$N_s = [l_1 1] \langle 14\rangle [4l_1] \langle l_1 l_3\rangle [l_3 3] \langle 32\rangle [2l_3] \langle l_3 l_1\rangle . \tag{46}$$

The evaluation of such products of spinors proceeds by using the spinor completeness relation [71] $\big|p\rangle [p] = \frac{1}{2}(1+\gamma_5)\, p$ and $\big| p]\langle p \big| = \frac{1}{2}(1-\gamma_5)\, p$, to turn N_s into $\big|$ a trace of products of Dirac matrices multiplied by momenta. Evaluating this trace through textbook methods and further using the on-shell and cut conditions leads to the simplified expression for the cut C_s in Eq. (45):

可以通过旋量完备性关系计算这类旋量乘积 $\big|p\rangle [p] = \frac{1}{2}(1+\gamma_5)\, p$ and $\big| p]\langle p \big| = \frac{1}{2}(1-\gamma_5)\, p$, to turn N_s into $\big|$ ，它是狄拉克矩阵与动量乘积的迹。通过经典方法计算该迹，再进一步利用在壳条件和截断条件，即可得到式 (45) 中截断 C_s 的简化表达式:

$$C_s = A_4^{\mathrm{tree}}(1,2,3,4) \frac{-ist}{(p-k_1)^2 (p+k_4)^2}. \tag{47}$$

Including the cut propagators, we arrive at the four-point one-loop color-ordered superamplitude,

计入截断传播子后，我们就得到了四点单圈色序超幅，

$$A_4^{1-\mathrm{loop}}(1,2,3,4) = -stA_4^{\mathrm{tree}} I_4(s,t), \tag{48}$$

where $I_4(s,t)$ is the scalar box integral,

其中 $I_4(s,t)$ 是标量箱积分，

$$I_4(s,t) = -i\int \frac{d^{4-2\varepsilon}p}{(2\pi)^{4-2\varepsilon}} \frac{1}{p^2 (p-k_1)^2 (p-k_1-k_2)^2 (p+k_4)^2}. \tag{49}$$

An important consistency condition is that the identical expression follows by analyzing the t -channel cut. It is quite remarkable that all integrals other than the scalar box integral cancel out, a fact that was understood long ago by considering the low-energy limit of string theory [47].

一个重要的一致性条件是，分析 t 道截断也会得到完全相同的表达式。非常值得注意的是，除标量箱积分外所有积分都相互抵消，这一结论早在弦论低能极限的研究中就已经得到了 [47]。

In the example discussed above, the only remaining dependence on Grassmann variables is in the overall delta function enforcing the conservation of the super-momentum. This structure is special to MHV super-amplitudes. In more general amplitudes, more fermionic variables remain after performing all Grassmann integrations. We note that the overall supermomentum delta function is sufficient to prove the finiteness of $\mathcal{N} \geq 2$ SYM amplitudes in four dimensions to all loop orders. For supergravity the overall Grassmann delta function is insufficient to prove finiteness because the dimensionful coupling constant implies an increasingly worse behavior in the absence of hidden symmetries.

在上述例子中，结果仅整体保留了约束超动量守恒的 delta 函数与格拉斯曼变量的依赖关系，这种结构是 MHV 超幅特有的。对于更一般的振幅，完成所有格拉斯曼积分后仍会剩余更多费米子变量。我们注意到，整体超动量 delta 函数足以证明四维下 $\mathcal{N} \geq 2$ SYM 振幅对所有圈阶都是有限的。对于超引力，整体格拉斯曼 delta 函数不足以证明有限性: 因为带量纲的耦合常数意味着，在不存在隐藏对称性的情况下，高圈阶的行为会越来越差。

The methods outlined and illustrated here have been used to construct amplitudes in $\mathcal{N} = 4$ SYM theory and $\mathcal{N} = 8$ supergravity through five loops [10-12,17,57, 165, 168, 170, 171, 173-175] as well as in a variety of other theories with less than maximal supersymmetry [107, 108, 112, 115]. The double copy played a central role in these calculations, either for simplifying the gravitational tree amplitudes that enter the cuts or for finding complete gravitational integrands as double copies of gauge-theory integrands.

本文概述并演示的方法已被用于构造五圈阶以内 $\mathcal{N} = 4$ SYM 理论和 $\mathcal{N} = 8$ 超引力的振幅 [10-12,17,57, 165, 168, 170, 171, 173-175]，也被用于多种超对称性小于最大超对称的其他理论 [107, 108, 112, 115]。双拷贝在这些计算中发挥了核心作用: 无论是简化进入截断的引力树图振幅，还是将完整引力被积函数构造为规范理论被积函数的双拷贝，都离不开它。

Loop-Level Duality Between Color and Kinematics

圈层次颜色与运动学对偶性

In section ”Color-Kinematics Duality and Double Copy” we discussed the duality between color and kinematics for tree-level amplitudes. Since the generalized unitarity method constructs loop integrands from tree amplitudes, it is natural to conjecture that the duality holds to all loop orders. While no formal proof has been constructed, many examples are available. Similar to the tree-level organization of amplitudes, any amplitude to any loop order can be written in terms of trivalent graphs; higher-point interaction vertices (or ”contact terms”) commonly present in Feynman diagrams are put in this form by multiplying and dividing by appropriate propagators. For any field theory with fields in the adjoint representation, L -loop m -point amplitudes can be written as

我们在“颜色-运动学对偶性与双重拷贝”一节中讨论了树级振幅的颜色与运动学对偶性。由于广义幺正方法是由树振幅构造圈积分被积函数，自然可以猜想该对偶性对所有圈阶都成立。目前虽未给出形式化证明，但已有大量实例验证。与树级振幅的组织方式类似，任意圈阶的任意振幅都可以用三价图表示；费曼图中常见的高点相互作用顶点 (或称“接触项”) 可以通过乘除适当传播子改写为这种形式。对于任何场属于伴随表示的场论，L 圈 m 点振幅可以写为

$$\mathcal{A}_m^{L-\text{loop}} = i^L g^{m-2+2L} \sum_{\mathcal{S}_m} \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j c_j}{\prod_{\alpha_j} p_{\alpha_j}^2}, \tag{50}$$

where second sum runs over all distinct m -point L -loop graphs with only trivalent vertices, labeled by j , the first sum runs over the set $\mathcal{S}_m$ of m ! permutations of external legs, and S_j is the symmetry factor of graph j and removes the overcount due to the automorphisms of this graph. For each term, there are LD -dimensional integrals over the independent loop momenta, and the denominator is given by the product of all (inverse) Feynman propagators of the corresponding graph. Similar to tree-level amplitudes, the color factor c_j is determined by the graph by associating to each vertex a gauge-group structure constant in a chosen (e.g., clockwise) labeling of lines. The kinematic numerators n_j are polynomials in Lorentz-invariant products of momenta, polarization vectors, and spinors and, for superamplitudes, also in Grassmann parameters. As at tree level, this amplitude is said to obey the duality between color and kinematics if the numerator factors have the same algebraic properties as those properties of the color factors - antisymmetry (17) and Jacobi relations (18) - that are required by gauge invariance. Generalized gauge transformations (22) are critical for obtaining kinematic numerators with these properties [4, 5, 100, 176] .

其中第二个求和遍历所有仅含三价顶点的不同 m 点 L 圈图，这些图由 j 标记，第一个求和对外部腿的 m ! 种置换构成的集合 $\mathcal{S}_m$ 进行，S_j 是图 j 的对称因子，用于消除图自同构导致的重复计数。对每一项，存在对独立圈动量的 LD 维积分，分母由对应图所有 (逆) 费曼传播子的乘积给出。与树级振幅类似，颜色因子 c_j 由图确定: 在选定的线标记 (例如顺时针标记) 下，给每个顶点分配一个规范群结构常数即可得到。运动学分子 n_j 是动量、极化矢量和旋量的洛伦兹不变乘积的多项式，对超振幅而言，还包含格拉斯曼参数。和树级情况一样，如果分子因子满足与颜色因子相同的代数性质——即规范不变性要求的反对称性 (17) 和雅可比关系 (18)，就称该振幅满足颜色与运动学对偶性。广义规范变换 (22) 对得到满足这些性质的运动学分子至关重要 [4, 5, 100, 176] 。

Although at the time of this writing the duality is a conjecture at loop level, it is supported by a number of nontrivial examples. In particular, in $\mathcal{N}=4$ SYM theory, superamplitudes manifestly obeying the duality have been constructed at four points through four loops [5,12], at five points at one and two loops [118,171], and at six and seven points at one loop [106,111,117,124]. In $\mathcal{N}=1$ and non-supersymmetric Yang-Mills theory, some four-point amplitudes obeying the duality at one and two loops may be found in, e.g., Refs. [5, 76, 105, 107, 112, 177]. Similarly, color-kinematics-satisfying representations of form factors in various supersymmetric gauge theories were found in, e.g., Refs. [114, 121, 122], and color-kinematics-satisfying representations of matrix elements of the open superstring effective field theory were discussed in, e.g., Ref. [123].

尽管截至本文写作时，该对偶性在圈层次仍是一个猜想，但它得到了许多非平凡实例的支持。特别是在 $\mathcal{N}=4$ 超对称杨-米尔斯理论中，满足对偶性的显式超振幅已经被构造出来: 四点到四圈 [5,12]，五点到一圈和两圈 [118,171]，六点和七点到一圈 [106,111,117,124]。在 $\mathcal{N}=1$ 和非超对称杨-米尔斯理论中，可以在参考文献 [5, 76, 105, 107, 112, 177] 等中找到满足该对偶性的一圈和两圈四点振幅。类似地，在多种超对称规范理论中，人们已经在参考文献 [114, 121, 122] 等中找到了满足颜色-运动学对偶性的形状因子表示，而开超弦有效场论矩阵元满足颜色-运动学对偶性的表示则在参考文献 [123] 等中被讨论。

At tree level, gauge-theory loop-level integrands manifestly obeying color-kinematics duality can be used to obtain gravitational integrands through the double-copy construction [4, 5] , that is, by substituting the color factors c_i of an integrand with the kinematic numerators n_i of another integrand, $c_i \to \tilde{n}_i$ (cf. Eq. (20)), at the same loop order and multiplicity. That is, at L -loops and for m external lines,

在树级，显式满足颜色-运动学对偶性的规范理论圈层次积分被积函数可以通过双重拷贝构造得到引力积分被积函数 [4, 5] ，具体来说，就是将一个被积函数的颜色因子 c_i 替换为另一个相同圈阶、相同多重度的被积函数的运动学分子 n_i ， $c_i \to \tilde{n}_i$ (参见式 (20))。也就是说，在 L 圈和 m 条外线的情况下，

$$\mathcal{M}_m^{L-\text{ loop}} = i^{L+1}\left(\frac{\kappa}{2}\right)^{m-2+2L} \sum_{S_m}\sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j \tilde{n}_j}{\prod_{\alpha_j} p_{\alpha_j}^2}. \tag{51}$$

Tree-level double copy guarantees that all generalized unitarity cuts of the resulting integrand reproduce the correct expression obtained by multiplying tree-level amplitudes and summing over intermediate states; cf. section ”Overview of the Unitarity Method.” For the same reason as at tree level, the double-copy formula (51) holds even if only one of the two sets of numerators, n_j or $\tilde{n}_j$, satisfies the duality manifestly. It is however necessary that it should be possible to bring the second set of numerators to a color-kinematics-satisfying form through generalized gauge transformations.

树级双复本保证了所得被积函数的所有广义幺正切割都能重现由树级振幅相乘并对中间态求和得到的正确表达式；参见“幺正方法概述”章节。和树级的情况同理，即便两组分子 n_j 或 $\tilde{n}_j$ 中仅有一组明显满足对偶性，双复本公式 (51) 依然成立。但必须满足的条件是，第二组分子能够通过广义规范变换变为满足色运动学对偶的形式。

Generalized Double Copy

广义双拷贝

Availability of gauge-theory integrands that satisfy color-kinematics duality manifestly guarantees a direct path to the corresponding gravitational integrands. The former are however not always straightforward to construct. In some cases such as the five-loop four-point $\mathcal{N}=8$ supergravity amplitudes, it has been proven difficult to find such a representation. In other cases, such as the all-plus two-loop five-gluon amplitude in pure Yang-Mills theory, the BCJ form of the amplitude has a superficial power count much worse than that of Feynman diagrams [112], and thus an analysis of ultraviolet properties of its double copy would more cumbersome than if the corresponding amplitudes were constructed directly from, e.g., generalized unitarity.

满足色运动学对偶性的规范理论被积函数的存在明确保证了可以直接得到对应的引力被积函数。但这类被积函数并不总是容易构造。在某些情况下，例如五圈四点 $\mathcal{N} = 8$ 超引力振幅，目前已经证明很难找到这样的表示形式。在另一些情况下，例如纯杨-米尔斯理论中的全正两 oop 五胶子振幅，该振幅的 BCJ 形式的表面幂次计数远差于费曼图的结果 [112]，因此分析其双拷贝的紫外性质会比通过广义幺正性等方法直接构造对应振幅更加繁琐。

A double-copy method that directly converts generic representations of gauge-theory amplitude to gravity ones was developed in Ref. [178] and applied in Refs. [17,174] to construct the five-loop four-point integrand of $\mathcal{N} = 8$ supergravity and study its ultraviolet properties after integration.

文献 [178] 提出了一种直接将规范理论振幅的一般表示转化为引力振幅的双拷贝方法, 文献 [17,174] 将其应用于构造 $\mathcal{N} = 8$ 超引力的五圈四点被积函数，并研究积分后的紫外性质。

The starting point is local representations of the two gauge-theory amplitudes that use only cubic graphs, as in Eq. (50). The next step is to naively carry out the double-copy substitution (20) to these amplitudes. In general, it does not result in a correct gravitational amplitude; nevertheless, this ”naive double copy” can be systematically improved to become the correct amplitude. The naive double-copy amplitude reproduces the maximal and next-to-maximal cuts of the expected (super)gravity amplitude whenever these cuts have only three-point and only one four-point tree-amplitude factor, respectively. This is because if it is in principle possible to construct color-kinematics-satisfying representation of gauge-theory amplitudes, then the three-point and four-point tree amplitudes manifestly obey the duality [4]. Beyond the next-to-maximal cuts, the naive double copy will generally not give correct unitarity cuts, and nontrivial corrections are necessary. These improvement terms are derived by finding the generalized gauge transformations that put the relevant trees entering the unitarity cuts in a color-kinematics-satisfying form and expressing them in terms of the (cut) numerators of the original gauge-theory amplitudes.

该方法的起点是仅使用三次图的两个规范理论振幅的局域表示，如式 (50) 所示。下一步是对这些振幅直接做双拷贝替换 (20)。一般而言，这不会得到正确的引力振幅；但这种“朴素双拷贝”可以通过系统修正得到正确的振幅。当预期 (超) 引力振幅的最大割仅包含三点树图振幅因子、次最大割仅包含一个四点树图振幅因子时，朴素双拷贝振幅可以精确重现这些切割结果。这是因为如果原则上可以构造满足色运动学对偶的规范理论振幅表示，那么三点和四点树振幅都显然满足该对偶性 [4]。超出次最大割之后，朴素双拷贝通常无法给出正确的幺正切割，因此需要引入非平凡修正。这些修正项通过寻找广义规范变换得到: 该变换将进入幺正切割的相关树图转化为满足色运动学对偶的形式，再将其用原规范理论振幅的 (切割) 分子项表示出来。

To illustrate these ideas, we discuss in some detail the generalized unitarity cut that decomposes a multiloop amplitude into sums over states of a product of three-point tree amplitudes and a single five-point amplitude, as in Fig. 5. This cut can be expressed as a sum over 15 diagrams, corresponding to the 15 diagrams with only cubic vertices that make up the 5-point tree amplitude,

为说明这些思路, 我们详细讨论将多圈振幅分解为三点树振幅乘积与单个五点树振幅乘积之和的广义幺正切割，如图 5 所示。该切割可以表示为 15 个图的和，对应构成五点树振幅的 15 个仅含三次顶点的图,

$$C_5 = \sum_{i=1}^{15} \frac{n_i \tilde{n}_i}{d_i^{(1)} d_i^{(2)}} + \mathcal{E}_5. \tag{52}$$

The first term is the naive double copy and the second is improvement term which turns out to be [174]

第一项是朴素双拷贝，第二项即为修正项，其结果为 [174]

$$\mathcal{E}_5 = -\sum_{i=1}^{15} \frac{\Delta_{i_1,i_2} \widetilde{\Delta}_{i_1,i_2}}{d_i^{(1)} d_i^{(2)}} = -\frac{1}{6} \sum_{i=1}^{15} \frac{J_{\{i,1\}} \tilde{J}_{\{i,2\}} + J_{\{i,2\}} \tilde{J}_{\{i,1\}}}{d_i^{(1)} d_i^{(2)}}, \tag{53}$$

where Δ and $\widetilde{\Delta}$ are the generalized gauge parameters that put the cut gauge-theory numerators in color-kinematics-satisfying form. The $J_{\{i,j\}}$ and $\tilde{J}_{\{i,j\}}$ are referred to as "BCJ discrepancy functions" and encode violations of the kinematic Jacobi relations, and the $1/d_i^{(j)}$ is the j th Feynman propagator of the i th diagram. Each $J_{i,j}$ consists of a sum, with appropriate signs, over the three numerator factors forming a Jacobi triplet, e.g.,

其中 Δ 和 $\widetilde{\Delta}$ 是广义规范参数，用于将切割后的规范理论分子项转化为满足色运动学对偶的形式。$J_{\{i,j\}}$ 和 $\tilde{J}_{\{i,j\}}$ 被称为"BCJ 差异函数"，用来描述运动学雅可比关系的破坏程度，$1/d_i^{(j)}$ 是第 j 个图的第 i 个费曼传播子。每个 $J_{i,j}$ 都是构成雅可比三元组的三个分子因子带适当符号的求和，例如，

$$J_{\{1,1\}} = n_1 + n_2 + n_3. \tag{54}$$

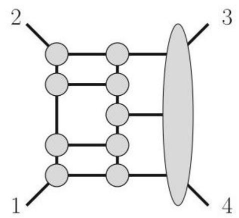


Fig. 5 A sample generalized cut decomposing the five-loop four-point amplitude into three-point amplitudes and a single five-point tree amplitude. Each blob represents a tree-level amplitude and the exposed lines are all on shell

图 5 一个示例广义切割，将五圈四点振幅分解为多个三点振幅和单个五点树振幅。每个团块代表一个树级振幅，暴露的线全部在壳

They vanish identically if the duality between color and kinematics is manifest.

如果色与运动学的对偶性是明显的，那么这些差异函数恒等于零。

In this way we can directly connect the corrections to the naive double copy to violations of the duality. See Ref. [174] for further details, including a scheme for labeling and tracking the appropriate triplets. Using such correction formulae, the naive double copy can be systematically promoted to a complete (super)gravity amplitude.

通过这种方式，我们可以将朴素双拷贝的修正直接与对偶性的破坏联系起来。更多细节包括标记和追踪对应三元组的方案可以参见文献 [174]。利用这类修正公式，朴素双拷贝可以系统地提升为完整的 (超) 引力振幅。

This procedure has been used to construct the complete five-loop four-point $\mathcal{N} = 8$ supergravity loop amplitude in Ref. [174]. The ultraviolet properties were described in Ref. [17] and are summarized in the next section.

文献 [174] 已经使用该方法构造了完整的五圈四点 $\mathcal{N} = 8$ 超引力圈振幅，其紫外性质在文献 [17] 中进行了阐述，我们将在下一节总结。

Comments on Regularization

正则化注释

Scattering amplitudes in general gauge and gravitational theories are both infrared and ultraviolet divergent and thus need to be regularized. Dimensional regularization, in which amplitudes are evaluated in $D = 4 - 2\varepsilon$ dimension, is preferred, including in massless theories [179], because of its flexibility and also because it preserves gauge invariance despite the care needed to preserve supersymmetry [180-182] (In planar $\mathcal{N} = 4$ SYM theory, only infrared divergences are present, and in this case a massive Higgs regulator has proven to be convenient because it preserves dual conformal invariance [183].). In this scheme, divergences appear as ε^{-n} for positive integers n , and because of their presence, the finite parts of amplitudes depend on terms in the integrand that are $\mathcal{O}(\varepsilon)$, as discussed in [154].

一般规范理论与引力理论中的散射振幅同时存在红外和紫外发散，因此需要进行正则化。将振幅放在 $D = 4 - 2\varepsilon$ 维中计算的维数正则化是更优的选择，这一点在无质量理论中同样成立 [179]，原因在于它的灵活性，且尽管需要格外注意才能保留超对称性 [180-182]，它始终保持规范不变性 (在平面 $\mathcal{N} = 4$ 超对称杨-米尔斯理论中，仅存在红外发散，这种情况下质量希格斯调节器已被证明十分方便，因为它保留了对偶共形不变性 [183])。在该方案中，发散表现为 ε^{-n} 的形式，其中 n 为正整数，正因为这些发散的存在，振幅的有限部分依赖于被积函数中为 $\mathcal{O}(\varepsilon)$ 的项，正如文献 [154] 所讨论的。

Dimensionally regularized one-loop scattering amplitudes in massless supersym-metric theories are fully determined by their generalized cuts in four dimensions [7]. This is directly related to their better power counting compared to their non-supersymmetric counterparts. In general, this no longer holds at higher loops, so the construction of the integrand must capture all terms to sufficiently high order in the dimensional regulator. Thus, integrands constructed from generalized cuts evaluated with four-dimensional helicity methods may need to be supplemented with further terms that vanish in four dimensions. Explicit calculations show that four-dimensional methods give the complete integrands for four-point amplitudes in $\mathcal{N} = 4$ SYM theory through six loops [9, 10, 92, 165, 169, 170, 173] , but not for higher-point amplitudes [184]. The double copy then implies that four-dimensional methods are sufficient for the corresponding four-point amplitudes in $\mathcal{N} = 8$ supergravity.

无质量超对称理论中经维数正则化的单圈散射振幅完全由其四维广义割线确定 [7]，这直接对应于相较于非超对称对应理论更好的幂计数性质。一般而言，这一性质在多圈情况下不再成立，因此构造被积函数时必须捕捉维数调节器中所有足够高阶的项。因此，通过四维螺旋度方法计算广义割线得到的被积函数，可能需要补充在四维下为零的额外项。显式计算表明，对于 $\mathcal{N}=4$ 超对称杨-米尔斯理论中直到六圈的四点振幅 [9, 10, 92, 165, 169, 170, 173]，四维方法可以给出完整被积函数，但对高点振幅不成立 [184]。由此通过双拷贝可得，四维方法对于 $\mathcal{N}=8$ 超引力中对应的四点振幅已经足够。

An interesting possibility is that all terms not determined by four-dimensional massless cuts have a predictable universal structure, at least near four dimensions. Until such a structure is proven and a strategy to evaluate these additional terms from four-dimensional data is devised, only D -dimensional calculations guarantee the completeness of the resulting amplitude. In general, using D -dimensional loop momenta [154, 185] makes calculations significantly more involved, because the powerful four-dimensional superspace [70,77,80,92,97] and spinor methods [186- 188] can no longer be used on the intermediate states in the cuts. This is mitigated by the six-dimensional helicity [81] and superspace [80] formalisms that we briefly reviewed in section ”Tree-Level Superamplitudes in Various Dimensions.” They have been used to directly confirm that the four-loop four-point integrand of $\mathcal{N}=4$ SYM and the associated $\mathcal{N}=8$ supergravity integrand obtained via double copy are consistent with unitarity in dimensional regularization [92].

一个值得关注的可能性是，所有未被四维无质量割线确定的项都具有可预测的普适结构，至少在接近四维的情况下成立。在证明这种结构存在、并设计出从四维数据计算这些额外项的方案之前，只有 D 维计算能保证所得振幅的完备性。一般而言，使用 D 维圈动量计算 [154, 185] 会复杂得多，因为强大的四维超空间 [70,77,80,92,97] 和旋量方法 [186-188] 无法再用于割线中的中间态。我们在“不同维度中的树-level 超振幅”一节简要回顾过的六维螺旋度 [81] 和超空间 [80] 形式体系缓解了这一问题。利用这些形式体系已经直接证实，通过双拷贝得到的 $\mathcal{N}=4$ 超对称杨-米尔斯四圈四点被积函数与对应的 $\mathcal{N}=8$ 超引力被积函数，符合维数正则化下的幺正性要求 [92]。

Ultraviolet Properties of Supergravity Theories

超引力理论的紫外性质

Symmetry Constraints on Counterterms

counter 项的对称性约束

The question of which counterterms satisfy all known symmetries is still a nontrivial question despite considerable effort to answer it. In the early 1980s, the consensus opinion was that the three loop R^4 was valid [47-49] (By assuming the existence of an extended superfield formalism that manifests all supersymmetries, one can raise this bound to seven loops [50]; later it was shown that the counterterm cannot be written as a full-superspace integral [22].). The restrictions supersymmetry and duality symmetry imposed on counterterms have been studied in great detail over the years.

尽管人们付出了大量努力，但究竟哪些 counter 项满足所有已知对称性仍然是一个尚待解决的问题。20 世纪 80 年代初，学界普遍认为三圈 R^4 是成立的 [47-49](通过假设存在一个能体现所有超对称性的扩展超场形式，该边界可被提升至七圈 [50]；后续研究证明，counter 项无法被写为全超空间积分 [22]。) 多年来，人们已经对超对称性和对偶对称性对 counter 项施加的限制进行了非常详尽的研究。

The three methods that have been applied to constrain the counterterms that can appear are as follows:

目前用于约束可出现 counter 项的共有三种方法，具体如下:

1. Use extended off-shell superspaces together with duality symmetries [18,18,20- 22, 28].

1. 结合对偶对称性使用扩展离壳超空间 [18,18,20-22,28]。

2. Use Berkovits' pure spinor formalism [58] to expose the full supersymmetry [19].

2. 使用 Berkovits 纯自旋形式 [58] 来体现完整超对称性 [19]。

3. Use maximal cuts [9] of amplitudes to expose the minimum powers of loop momenta required in any covariant Feynman-like diagrams [16,63]. This predicts where divergences appear, assuming no further cancellations between diagrams.

3. 使用振幅的极大切割 [9]，确定任意协变费曼类图所需的圈动量最小幂次 [16,63]。该方法在假设不同图之间不存在额外抵消的前提下，可预言发散的出现位置。

These three methods lead to the same constraints on counterterms for the various theories displayed in Table 2. The last of these methods implies that cancellations that occur beyond this cannot be manifested diagram by diagram. By definition, any further cancellations beyond the ones exposed by the above methods are enhanced ultraviolet cancellations [16]. Table 2 displays the first available counterterms that satisfy all known symmetry constraints for various theories of interest. In the table, D is the space-time dimensions and Q is the number of supercharges.

对于表 2 列出的多种理论，这三种方法给出的 counter 项约束完全一致。其中最后一种方法指出，超出该范围的抵消无法逐图体现。根据定义，所有超出上述方法已揭示抵消的额外抵消都属于增强紫外抵消 [16]。表 2 列出了各类目标理论中满足所有已知对称性约束的首个可用 counter 项。表中，D 为时空维度，Q 为超荷数。

For $\mathcal{N} = 8$ supergravity in $D = 4$, as demonstrated in various studies, a counterterm is allowed at $L = 7$ loops [18,18-21]. By reducing the number of supercharges, the loop order where the first viable counterterm exists is lowered. In particular, in $D = 4\mathcal{N} = 4$ supergravity, the first counterterm is allowed at $L = 3$ loops. For $\mathcal{N} = 5$ supergravity the loop order is raised to $L = 4$ loops [22, 28] . As explained in Ref. [22], these counterterms cannot be written as full-superspace integrals, but they do appear to respect all known standard-symmetry considerations. By increasing the space-time dimensions, one can also lower the loop order at which a divergence can first appear. For example, half-maximal 16-supercharge supergravity in $D = 5$ exhibits a possible two-loop counterterm invariant under all known symmetries [60, 60] . By taking an unphysical dimension of $D = 24/5$ for maximal 32-supercharge supergravity, corresponding to $\mathcal{N} = 8$ supergravity in

$D = 4$, the loop order where a divergence first appears is lowered from $L = 7$ to $L = 5$ [19]. See Refs. [14, 60, 64, 65, 189, 190] for various attempts to put tighter restrictions on the counterterms using symmetry, including double-copy consistency, and the associated difficulties.

对于 $\mathcal{N} = 8$ 维 $D = 4$ 超引力，多项研究表明，在 $L = 7$ 圈允许存在 counter 项 [18,18-21]。降低超荷数会使得首个可行 counter 项出现的圈阶降低。具体而言，在 $D = 4\mathcal{N} = 4$ 超引力中，首个 counter 项允许出现在 $L = 3$ 圈。对于 $\mathcal{N} = 5$ 超引力，该圈阶提升至 $L = 4$ 圈 [22, 28] 。正如文献 [22] 所述，这些 counter 项无法被写为全超空间积分，但它们确实满足所有已知标准对称性的要求。增加时空维度同样会降低首个发散出现的圈阶。例如，$D = 5$ 维半最大 16 超荷超引力中，可能存在一个满足所有已知对称性不变性的两圈 counter 项 [60, 60] 。若取最大 32 超荷超引力的非物理维度为 $D = 24/5$ ，对应 $\mathcal{N} = 8$ 维 $D = 4$ 超引力，那么首个发散出现的圈阶会从 $L = 7$ 降至 $L = 5$ [19]。关于各类利用对称性对 counter 项施加更严格约束的尝试 (包括双对拷贝一致性在内) 以及相关难点，参见文献 [14, 60, 64, 65, 189, 190] 。

Table 2 Counterterms corresponding to the first potential divergence that satisfy all proven super-symmetry and duality-symmetry constraints [18, 19, 21, 22, 28, 60] . The number of supercharges is Q and D is the space-time dimension

表 2 满足所有已证明超对称性和对偶对称性约束的首个潜在发散对应的 counter 项 [18, 19, 21, 22, 28, 60] 。超荷数为 Q ，D 为时空维度

Theory	Counterterm	Loop order	Divergence
$D = 4, Q = 32, \mathcal{N} = 8$	$\mathcal{D}^8R^4$	7	Unknown
$D = 4, Q = 16, \mathcal{N} = 4$	R^4	3	No
$D = 4, Q = 20, \mathcal{N} = 5$	$\mathcal{D}^2R^4$	4	No
$D = 24/5, Q = 32$	$\mathcal{D}^8R^4$	5	Yes
$D = 5, Q = 16$	R^4	2	No

Explicit Calculations of Counterterm Coefficients

抵消项系数的显式计算

A central question is whether the potential counterterms in Table 2 lead to actual divergences in the respective theories. The systematic way to settle this is to carry out direct computations in a way that avoids potential subtleties or unproven arguments. We first describe the status of maximal $\mathcal{N} = 8$ supergravity in four space-time dimensions before turning to cases with less supersymmetry. As discussed in section "Loop-Level Methods," the unitarity method [7-9] in conjunction with the double copy [$3 - 5$, 178] is used to construct amplitudes' integrands, from which the ultraviolet divergences are extracted. This gives us the coefficient of any potential counterterm, exposing any hidden cancellations [10-12, 17, 57, 173]. String-theory calculations also provide important information [18,47,59]. Calculations demonstrating the power counting of unitarity cuts [23, 24, 191, 192] also provide important insight. Here we will review the results of such computations and emphasize various cancellations that are unexplained by symmetry.

核心问题在于表 2 中的潜在抵消项是否会在对应理论中引发真实发散。解决该问题的系统方法是开展直接计算，避免潜在的微妙问题或未经证明的论断。在讨论超对称性更低的情形之前，我们首先介绍四维最大 $\mathcal{N} = 8$ 超引力的研究现状。正如“圈图层次方法”一节所述，幺正性方法 [7-9] 结合双拷贝 $[3-5, 178]$ 可用于构造振幅被积函数，从中提取紫外发散。这一方法能给出任意潜在抵消项的系数，揭示所有隐藏抵消 [10-12, 17, 57, 173]。弦理论计算也能提供重要信息 [18,47,59]。证明幺正切割幂次计数的计算 [23, 24, 191, 192] 同样带来了重要启发。本文将综述这类计算的结果，重点介绍无法通过对称性解释的各类抵消。

In Table 3 we collect the critical dimensions where explicit calculations show that maximally supersymmetric supergravity first diverges. In contrast to expectations in the early 1980s, these calculations show that $\mathcal{N} = 8$ supergravity is finite through at least five loops. As described above, these ultraviolet cancellations were subsequently understood to follow from supersymmetry and the $E_{7(7)}$ duality symmetry of $\mathcal{N} = 8$ supergravity, with the first $D = 4$ potential divergence delayed until seven-loop order.

表 3 汇总了显式计算得到的最大超引力首次出现发散的临界维度。与 20 世纪 80 年代初的预期不同，这些计算表明 $\mathcal{N} = 8$ 超引力至少直到五圈都是有限的。如上所述，这类紫外抵消后来被证明可以从超对称性以及 $\mathcal{N} = 8$ 超引力的 $E_{7(7)}$ 对偶对称性导出，首个 $D = 4$ 潜在发散被推迟到七圈阶才会出现。

To carry out such calculations, one first constructs an integrand in $\mathcal{N} = 4$ SYM theory and then promote it through double copy to an $\mathcal{N} = 8$ integrand

开展这类计算时，我们首先在 $\mathcal{N} = 4$ SYM 理论中构造被积函数，再通过双拷贝将其提升为 $\mathcal{N} = 8$ 被积函数

$$(\mathcal{N} = 8 \text{ supergravity}) \sim (\mathcal{N} = 4\text{SYM}) \times (\mathcal{N} = 4\text{SYM})\,. \tag{55}$$

The integrand obtained in this way is then expanded for large loop momenta or equivalently for small external momenta in order to extract the ultraviolet divergences (At five loops this process is more complicated because of difficulties of finding five loop integrands that satisfy color-kinematics duality; in this case a generalized double-copy procedure is instead applied [174, 178].). This expansion gives a set of vacuum-like diagrams that can be reduced to a small number of ”basis integrals” using integration by parts identities [193]. The remaining basis integrals are usually simple enough to be analytically integrated giving the exact value of the ultraviolet divergence.

随后对由此得到的被积函数在大圈动量 (等价于小外动量) 下展开，以提取紫外发散 (五圈时该过程更复杂，因为难以找到满足色-动对偶的五圈被积函数；这种情况下会改用广义双拷贝方法 [174, 178])。展开后会得到一组类真空图，利用分部积分恒等式可将其约化为少量“基积分”[193]。剩余的基积分通常足够简单，可以解析积分得到紫外发散的精确值。

Table 3 The critical dimensions D_c where ultraviolet divergences first occur in maximal $\mathcal{N} = 8$ supergravity, as proven by explicit calculations [10- 12, 17, 57, 173]

表 3 显式计算证明的最大 $\mathcal{N} = 8$ 超引力中紫外发散首次出现的临界维度 D_c [10- 12, 17, 57, 173]

Loops	Critical dimension
1	8
2	7
3	6
4	11/2
5	24/5

After reducing the vacuum-like integrals, we obtain a simple description of the leading ultraviolet behavior in terms of a basis of vacuum integrals defined as

约化类真空积分后，我们可以利用定义如下的真空积分基得到领头紫外行为的简洁描述:

$$V = -i^{L+\sum_j A_j} \int \prod_{i=1}^{L} \frac{d^D \ell_i}{(2\pi)^D} \prod_j \frac{1}{\left(p_j^2 - m^2\right)^{A_j}}, \tag{56}$$

where the p_i are linear combinations of the independent loop momenta and the A_i are the propagators' exponents. To make the integrals well defined, we use dimensional regularization $D = D_c - 2\varepsilon$ where D_c is the critical dimension where the first divergence appears. In dimensional regularization any divergence will appear as a $1/\varepsilon$. It is useful to introduce a mass as an infrared regulator to make it easier to separate out infrared singularities from the ultraviolet ones.

其中 p_i 是独立圈动量的线性组合，A_i 是传播子的指数。为了让积分良定义，我们采用维数正规化 $D = D_c - 2\varepsilon$ ，其中 D_c 是首个发散出现的临界维度。在维数正规化中，所有发散都会表现为 $1/\varepsilon$ 。引入质量作为红外调节器有助于分离红外奇点和紫外奇点。

Collecting the results from Refs. [10-12, 17, 47, 57, 173], the leading ultraviolet behavior of $\mathcal{N} = 8$ supergravity at each loop order through five loops is described by vacuum diagrams as, where the vacuum integrals are to be evaluated using dimensional regularization around the critical dimension listed in Table 3,

汇总文献 [10-12, 17, 47, 57, 173] 的结果，直到五圈，$\mathcal{N} = 8$ 超引力各圈阶的领头紫外行为都可以用真空图描述，其中真空积分需要在临界维度附近通过维数正规化计算，临界维度已列在表 3 中，

$$\mathcal{M}_4^{(1)}\Big|_{\text{leading}}^{D_c=8} = -3\mathcal{K}_{\text{G}}\left(\frac{\kappa}{2}\right)^4 (57)$$

$$\mathcal{M}_4^{(2)}\Big|_{\text{leading}}^{D_c=7} = -8\mathcal{K}_{\text{G}}\left(\frac{\kappa}{2}\right)^6 (s^2+t^2+u^2)\left(\frac{1}{4}\left(1+\frac{1}{4}+\frac{1}{4}+\cdots\right)\right),$$

$$\mathcal{M}_4^{(3)}\Big|_{\text{leading}}^{D_c=6} = -60\mathcal{K}_{\text{G}}\left(\frac{\kappa}{2}\right)^8 \text{stu}\left(\frac{1}{6}(2-3)+\frac{1}{2}(2-3)\right),$$

$$\mathcal{M}_4^{(4)}\Big|_{\text{leading}}^{D_c=11/2} = -\frac{23}{2}\mathcal{K}_{\text{G}}\left(\frac{\kappa}{2}\right)^{10} (s^2+t^2+u^2)^2\left(\frac{1}{4}(1-s)+\frac{1}{2}(1-s)+\frac{1}{4}(1-s)\right),$$

$$\mathcal{M}_4^{(5)}\Big|_{\text{leading}}^{D_c=24/5} = -\frac{16\times 629}{25}\mathcal{K}_{\text{G}}\left(\frac{\kappa}{2}\right)^{12} (s^2+t^2+u^2)^2\left(\frac{1}{48}(1-s)+\frac{1}{16}(1-u)\right),$$

where the prefactor is proportional to the tree amplitude, $\mathcal{K}_{\text{G}} \equiv \text{stu}\, M_4^{\text{tree}}\ (1,2,3,4)$. The lines represent propagators appearing in Eq. (56), and the number of dots j on propagator corresponds to the $A_j - 1$ for $A_j \geq 2$ where A_j is the index appearing in Eq. (56). Explicit values of the integrals are found in the source papers, but for our purposes here, it is sufficient to note that the basis integrals are all divergent with a definite sign so no further cancellations are possible.

其中前因子正比于树图振幅 $\mathcal{K}_{\mathrm{G}} \equiv \mathrm{stu}\, M_4^{\mathrm{tree}}(1,2,3,4)$ 。图中的线代表式 (56) 中出现的传播子，传播子上的点数 j 对应 $A_j - 1$ ，用于 $A_j \geq 2$ ，其中 A_j 是式 (56) 中出现的指标。积分的明确取值可在原始文献中找到，但就我们此处的目的而言，只需注意基积分全部带有确定符号发散，因此不可能发生进一步抵消就足够了。

Given the wealth of information contained in Eq. (57), one might wonder if there is a path to finding a shortcut to the results, or perhaps even to write down the exact divergence in the critical dimension in terms of vacuum integrals to all loop orders. An attempt in this direction is found in Ref. [17]. Unfortunately, the constraints from factorization-type arguments do not appear to be sufficient to uniquely determine the vacuum diagrams at higher orders. Nevertheless, with additional information it might still be possible to make an educated guess for the exact form of the divergences in $\mathcal{N}=8$ supergravity in the critical dimensions.

考虑到式 (57) 包含的丰富信息，人们或许会好奇，是否存在捷径得到结果，甚至能否将临界维度下的精确发散用真空积分写出到所有圈阶。文献 [17] 已经做了这方面的尝试。遗憾的是，因子化类论证给出的约束似乎不足以唯一确定高阶的真空图。尽管如此，借助额外信息，我们仍有可能对临界维度下 $\mathcal{N}=8$ 超引力中发散的精确形式做出合理猜测。

If the supersymmetry is reduced, the potential counterterms become more complicated, but the loop order where they might first occur is lowered as well. The case of $N=4$ supergravity is particularly interesting. As discussed in previous sections, via the double copy, pure $\mathcal{N}=4$ supergravity can be decomposed into gauge theories,

如果超对称破缺，潜在抵消项会变得更复杂，但它们首次出现的圈阶也会降低。$N=4$ 超引力的情形尤其值得关注。正如之前章节讨论的，通过双拷贝，纯 $\mathcal{N}=4$ 超引力可以分解为规范理论，

$$(\mathcal{N}=4 \text{ supergravity }) \sim (\text{ pure YM }) \times (\mathcal{N}=4 \mathrm{SYM}). \tag{58}$$

While it does not have a divergence at the three loops [194] contrary to symmetry expectations [47-49], it, however, does diverge at four loops as proven by explicit calculation [15]. The counterterm can be expressed in terms of four powers of the Riemann tensor with the explicit form,

与对称性预期 [47-49] 相反，它在三圈没有发散 [194]，但正如显式计算 [15] 证明的，它在四圈确实存在发散。抵消项可以用黎曼张量的四次方表示，其显式形式为

$$C^{L=4, \mathcal{N}=4} = -\frac{1}{(4\pi)^8}\left(\frac{\kappa}{2}\right)^6 \frac{1}{72\varepsilon}\left(1-264\zeta_3\right)\left(T_1+2T_2\right), \tag{59}$$

where

其中

$$T_1 \equiv \left(D_\alpha R_{\mu\nu\lambda\gamma}\right)\left(D^\alpha R_{\rho\sigma}{}^{\lambda\gamma}\right) R^{\nu\rho}_{\delta\kappa} R^{\sigma\mu\delta\kappa},$$

$$T_2 \equiv \left(D_\alpha R_{\mu\nu\lambda\gamma}\right)\left(D^\alpha R_{\rho\sigma}{}^{\lambda\gamma}\right) R^{\mu\nu}_{\delta\kappa} R^{\rho\sigma\delta\kappa}. \tag{60}$$

This expression drops evanescent contributions that vanish in strictly four dimensions and are nonsingular in $D = 4 - 2\varepsilon$ with ε small. Using the helicity form of the divergence calculated in Ref. [15], one can also obtain expressions for the counterterms of other states. Indeed, superspace forms of such operators are discussed in Refs. [62, 195-197].

该表达式舍去了倏忽贡献，这类贡献在严格四维中为零，且在 $D = 4 - 2\varepsilon$ 当 ε 很小时非奇异。利用文献 [15] 中计算得到的发散的螺旋度形式，我们还可以得到其他态的抵消项表达式。实际上，文献 [62, 195-197] 已经讨论了这类算符的超空间形式。

An important question is how the four-loop divergence of pure $\mathcal{N} = 4$ super-gravity (59) should be interpreted. A complication is that this theory has a rigid $U(1)$ duality-symmetry anomaly [61, 62] . Interestingly, the helicity dependence of contributions of the unitarity cuts [62] and the divergence [14] suggests that the divergence is tied to the $U(1)$ duality anomaly of the theory. Without the anomaly, the - + + + and + + + + helicity sectors would vanish. Ref. [62] demonstrates that the anomaly leads to a poor ultraviolet behavior in the $--++$ as well. In this way the divergence is directly connected to the appearance of anomalous amplitudes which vanish at tree level but are nonzero starting at one loop [198]. It would be important to demonstrate this directly, presumably by tracking the anomaly contributions to four-loop amplitudes and showing that all but anomaly terms cancel. One may also wonder whether it is possible to remove the divergence by adding a finite term to the action so that an appropriate symmetry required for finiteness is preserved [25]. This brings up the question of whether the pure supergravity theories, which do not have such anomalies, are ultraviolet-finite or at least have an improved behavior [15, 25, 26] .

一个重要问题是，如何理解纯 $\mathcal{N} = 4$ 超引力 (59) 的四圈发散。复杂之处在于，该理论存在刚性 $U(1)$ 对偶对称反常 [61, 62] 。有趣的是，幺正切割贡献 [62] 和发散 [14] 的螺旋度依赖表明，发散与该理论的 $U(1)$ 对偶反常绑定。若没有反常，- + + + 和 + + + + 螺旋度扇区都会为零。文献 [62] 证明，反常也会导致 $--++$ 中紫外行为变差。因此，发散直接与反常振幅的出现相关，这类振幅在树图阶为零，但从一圈开始就非零 [198]。直接证明这一点十分重要，大概可以通过追踪反常对四圈振幅的贡献，证明除反常项外所有项都抵消来完成。人们也会好奇，能否通过给作用量添加有限项来消去发散，同时保持有限性所需的适当对称性 [25]。这就引出了一个问题: 不带有这类反常的纯超引力理论，是否是紫外有限的，或至少行为得到改善 [15, 25, 26] 。

In $\mathcal{N} = 5$ supergravity in $D = 4$, there are no anomalous amplitudes of the type that appear in $\mathcal{N} = 4$ supergravity [27] making it an ideal case to investigate. From the double-copy perspective, this theory is given as a product of

在 $D = 4$ 中的 $\mathcal{N} = 5$ 超引力里，不存在 $\mathcal{N} = 4$ 超引力中出现的那类反常振幅 [27]，这使得它成为理想的研究对象。从双拷贝的视角看，该理论可以表示为以下乘积

$$(\mathcal{N} = 5 \text{ supergravity }) \sim (\mathcal{N} = 1\text{SYM}) \times (\mathcal{N} = 4\text{SYM}) . \tag{61}$$

The computation of the potential four-loop divergence in this theory follows closely the one of $\mathcal{N} = 4$ supergravity in four dimensions. In this case, the potential counterterm vanishes [16],

该理论中潜在四圈发散的计算与四维下 $\mathcal{N}=4$ 超引力的计算高度相似。在这个情形中，潜在抵消项为零 [16]，

$$C^{L=4,N=5}=0\,. \tag{62}$$

This vanishing is nontrivially involving cancellations between planar and nonplanar contributions. The lack of a robust symmetry argument for this vanishing has been confirmed in Ref. [28].

该消失非平庸地涉及平面贡献和非平面贡献之间的抵消。文献 [28] 已经证实，目前不存在针对该消失的可靠对称性论证。

A particularly striking example of a theory exhibiting enhanced cancellations is pure half-maximal supergravity in $D=5$. The first potential counterterm is two loops, corresponding to an R^4 counterterm. The double-copy construction for this case is

展现出增强抵消的一个特别引人注目的例子是 $D=5$ 中的纯半最大超引力。第一个潜在抵消项出现在两圈，对应一个 R^4 抵消项。该情形的双复构造为

$$\text{(half-maximal supergravity)} \sim (\text{ pure YM }) \times (\text{ maximal SYM })\,, \tag{63}$$

which is essentially the same one as for $\mathcal{N}=4$ supergravity in $D=4$ (58), except that the two-component gauge theories are promoted to $D=5$. This case is simple enough so that the cancellations can be analyzed in detail [13]. By assuming the existence of an off-shell superspace that manifests all supersymmetries, such a counterterm can be ruled out [60]; however, doubt is cast on the existence of such a superspace by it also predicting finiteness for certain matter-coupled supergravities for which explicit divergences were found [14].

它本质上和 (58) 式中 $D=4$ 下 $\mathcal{N}=4$ 超引力的构造相同，区别仅在于两分量规范理论被提升为 $D=5$ 。这个情形足够简单，因此可以详细分析抵消过程 [13]。若假设存在一个能体现所有超对称性的脱壳超空间，就可以排除这类抵消项 [60]；然而，该超空间还预言了一些耦合物质的超引力是有限的，但人们已经在这些超引力中发现了明显发散 [14]，因此这类超空间的存在性受到了质疑。

Despite the lack of a symmetry argument, the coefficient of the potential $L=2$, $D=5$ divergence in pure half-maximal supergravity does indeed vanish [13]. It is noteworthy that the absence of divergences [59] in half-maximal supergravity in $D=5$ has also been obtained from string-theory calculations, offering an alternative approach to expose the ultraviolet cancellations in this theory.

尽管缺乏对称性论证，纯半最大超引力中潜在的 $L=2$ 、$D=5$ 发散的系数确实为零 [13]。值得注意的是，弦理论计算也得到了 $D=5$ 中半最大超引力不存在发散的结论 [59]，这为揭示该理论中的紫外抵消提供了另一种方法。

Remarkably, the enhanced cancellations for this example are directly tied to the double-copy structure inherent in the amplitudes [13]. The diagrams that result from the double copy are individually ultraviolet divergent with quadratic divergences appearing at two loops. Nevertheless, when all diagrams are combined, the divergences cancel from the amplitude [13]. This case is particularly simple to analyze because of special properties of the two-loop maximal SYM four- and five-point amplitudes. These amplitudes carry no powers of loop momentum in the kinematic numerators, making it straightforward to directly express the supergravity amplitudes after loop integration directly in terms of integrated pure Yang-Mills amplitudes. This allows us to directly relate two-loop four- and five-point potential divergences in supergravity to corresponding ones of pure Yang-Mills theory. Gauge invariance dictates the form of the potential counterterms in Yang-Mills theory, putting restrictions on the color factors that can appear. Through the double copy, restrictions on gauge-theory divergences then imply restrictions on the supergravity divergences. In this way finiteness of half-maximal supergravity four- and five-point amplitudes at one loop in $D < 8$ and at two loops in $D < 6$ can be understood, with the cancellations being locked to those of well-understood forbidden color factors in corresponding pure Yang-Mills amplitudes. Unfortunately, it is not straightforward to extend this analysis to higher loops because the integrals one encounters are no longer identical to the ones encountered in gauge theory. Nevertheless, this does demonstrate the importance of hidden structures of symmetries severely limiting the form of supergravity divergences.

值得注意的是，这个例子中的增强抵消与振幅固有的双复结构直接相关 [13]。双复得到的单个图都是紫外发散的，两圈处会出现二次发散。但当所有图合并后，发散会从振幅中抵消掉 [13]。这个情形分析起来格外简单，因为两圈最大 SYM 四点和五点振幅具有特殊性质。这些振幅的运动学分子不携带圈动量幂次，因此在圈积分后，可以很方便地直接将超引力振幅用积分后的纯杨-米尔斯振幅表示。这让我们可以直接将超引力中两圈四点和五点的潜在发散与纯杨-米尔斯理论中对应的发散联系起来。规范不变性决定了杨-米尔斯理论中潜在抵消项的形式，对可能出现的颜色因子给出了限制。通过双复，规范理论发散受到的限制进一步转化为超引力发散受到的限制。通过这种方式，我们可以理解半最大超引力的四点、五点振幅在 $D < 8$ 单圈和 $D < 6$ 两圈处的有限性，其抵消与纯杨-米尔斯振幅中已被充分理解的禁戒颜色因子的抵消严格对应。遗憾的是，很难将该分析推广到更高圈，因为此时遇到的积分不再和规范理论中的积分相同。尽管如此，这依然证明了隐藏的对称性结构对超引力发散形式的强限制作用十分重要。

Web of Supergravity Theories

超引力理论网络

Up to this point, we have studied the double-copy structure and ultraviolet properties mostly within the context of $\mathcal{N} \geq 4$ supergravities. The reader may wonder whether similar progress can be made for other classes of gravity theories.

到目前为止，我们大多是在 $\mathcal{N} \geq 4$ 超引力的框架下研究双拷贝结构与紫外性质。读者或许会好奇，其他类别的引力理论能否取得类似进展。

As we shall briefly discuss in this section, there is by now extensive evidence that very large families of gravitational theories can be seen as double copies. However, at present, it is not clear whether all gravities are double-copy constructible and, if not, what are the criteria that determine whether a double copy can be formulated for a given theory. In Fig. 6, we give a pictorial rendition of the web of theories linked by common gauge-theory factors in their double-copy constructions. While there is not sufficient space here to discuss all individual nodes, we briefly comment on some examples. For further details, the reader is invited to study the reviews [6] and [157].

我们将在本节简要讨论，目前已有充分证据表明，极大类引力理论都可以被视作双拷贝。但目前尚不明确是否所有引力都可以通过双拷贝构造，若不能，又该依据什么标准判断给定理论能否构造出双拷贝。在图 6 中，我们对双拷贝构造中通过共同规范理论因子连接的理论网络做了图示呈现。限于篇幅，我们在此不逐一讨论每个节点，仅对部分例子做简要说明。读者可参阅综述文献 [6] 和 [157] 了解更多细节。

Purely Adjoint Double Copies

纯伴随双拷贝

The fundamental ingredient entering a double-copy construction is the set of duality-satisfying numerators from a gauge theory. We have also seen that both SYM theory with various amounts of supersymmetry and pure Yang-Mills theory in any dimension satisfy the duality. In addition to that, if we restrict our analysis to gauge theories with only adjoint fields, it is relatively straightforward to find additional simple examples of theories that obey color-kinematics duality:

双拷贝构造的基础要素是规范理论中满足对偶性的分子集合。我们已经知道，不同超对称量的超杨-米尔斯理论与任意维度的纯杨-米尔斯理论都满足该对偶性。除此之外，如果我们仅研究仅含伴随场的规范理论，可以很容易找到更多满足颜色-运动学对偶性的简单理论例子:

- One can consider consistent truncations (sometimes called field-theory orbifolds) of $\mathcal{N} = 4$ SYM theory that preserve a portion of the original supersymmetry and only have adjoint fields (see Refs. [76, 105, 199] for a comprehensive discussion).

- 可以对 $\mathcal{N} = 4$ SYM 理论做一致截断 (有时也称为场论轨形)，这类截断保留部分原始超对称，且仅包含伴随场 (全面讨论见文献 [76, 105, 199])。

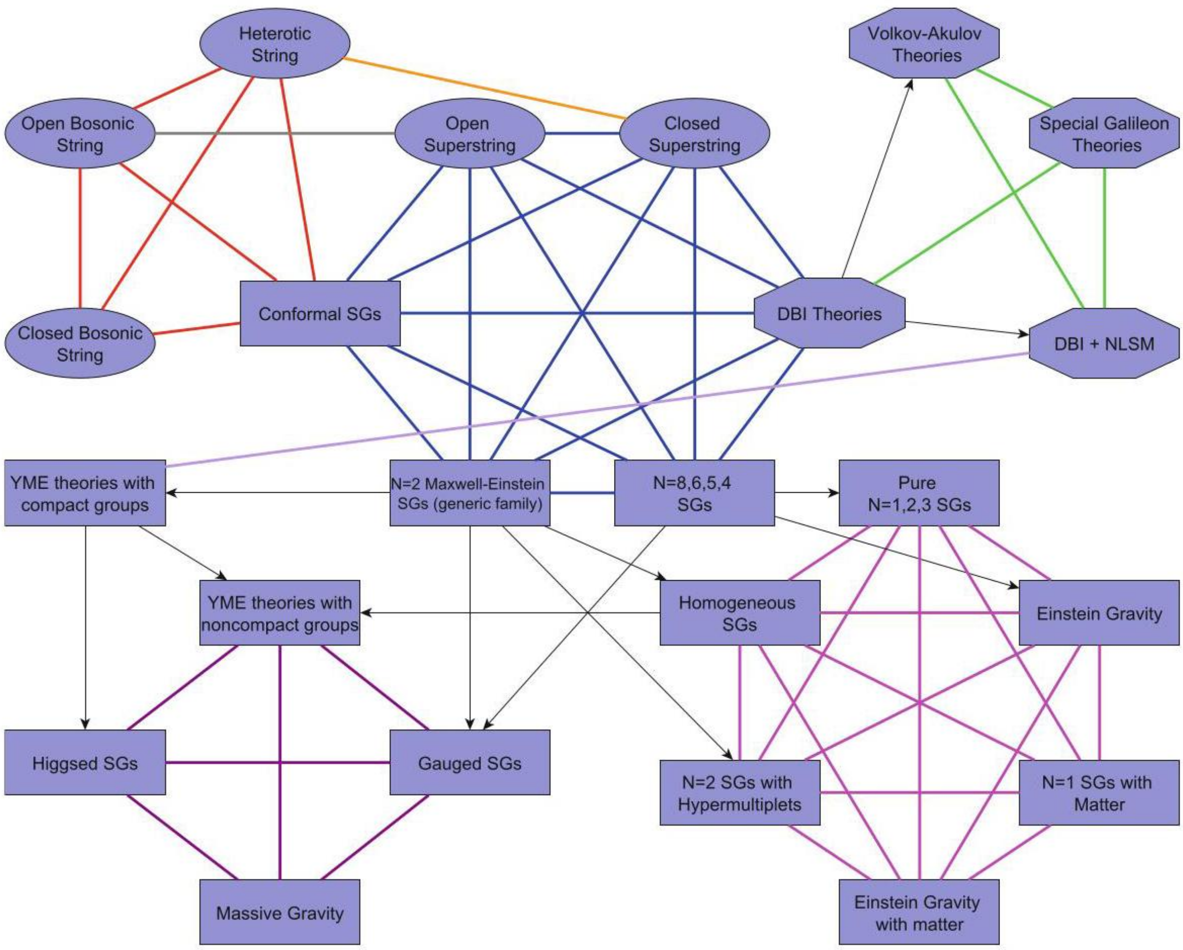


Fig. 6 Web of double-copy-constructible theories. Links with different colors connect theories that have a common gauge theory in their construction. Directed links point toward constructions that are obtained by modifying both gauge-theory factors (see Refs. [6, 157]). The main text provides a discussion of theories on/close to the central hexagon

图 6 可通过双拷贝构造的理论关系网。不同颜色的连线连接构造中包含共同规范理论的理论。有向连线指向通过同时修改两个规范理论因子得到的构造 (见文献 [6, 157])。正文讨论中心六边形及临近位置的理论

They yield, aside from pure $\mathcal{N}=1,2$ SYM theories, also $\mathcal{N}=1,2$ SYM with one vector multiplet.

除纯 $\mathcal{N}=1,2$ SYM 理论外，这类截断还得到带有一个矢量多重态的 $\mathcal{N}=1,2$ SYM 理论。

- The dimensional reduction of a pure Yang-Mills theory in D -dimensions to d dimensions gives a Yang-Mills-scalar theory with adjoint scalars and a $SO(D-$ d) global symmetry. Note that, in general, dimensional reduction preserves color-kinematics duality.

- 将 D 维纯杨-米尔斯理论维数约化到 d 维，可得到带有伴随标量和 $SO(D-$ 维整体对称性的杨-米尔斯-标量理论。注意，一般来说维数约化保持颜色-运动学对偶性。

- The latter theory can be deformed introducing trilinear scalar couplings of the form [108]

- 上述后者理论可以引入如下形式的三线性标量耦合进行形变 [108]

$$\delta\mathcal{L} = \frac{\lambda}{6!} F^{IJK} \operatorname{Tr}\left[\phi^I, \phi^J\right] \phi^K, \tag{64}$$

where F^{IJK} are structure constants for a subgroup of the $SO(D-d)$ global symmetry, I, J, K are global indices carried by the scalars, and λ is a free parameter. In this case, color-kinematics duality implies that the F^{IJK} tensors obey Jacobi relations.

其中 F^{IJK} 是 $SO(D-d)$ 整体对称性一个子群的结构常数，I, J, K 是标量携带的整体指标，λ 是自由参数。在此情形下，颜色-运动学对偶性要求 F^{IJK} 张量满足雅可比关系。

- The nonlinear sigma model (NLSM) can also be shown to obey color-kinematics duality, with the caveat that in this case the color gauge group is replaced by a global (flavor) group [130-132, 134, 139, 200-202]. The NLSM is a sufficiently simple example that a Lagrangian manifesting the duality is available; see Ref. [139].

- 非线性 sigma 模型 (NLSM) 也可被证明满足颜色-运动学对偶性，但需要注意该模型中色规范群被整体 (味) 群取代 [130-132, 134, 139, 200-202]。非线性 sigma 模型足够简单，存在显式体现对偶性的拉格朗日量；见文献 [139]。

Aside from the above cases, there is another known gauge theory with only adjoint fields that obeys color-kinematics duality: a higher-derivative version of Yang-Mills theory with a mass parameter that has been dubbed $(DF)^2$ theory [190,203,204]. While there exist several distinct versions, the simplest (minimal) theory has Lagrangian,

除上述情形外，目前已知还有一个仅含伴随场且满足颜色-运动学对偶性的规范理论: 带质量参数的高阶导数版本杨-米尔斯理论，被称为 $(DF)^2$ 理论 [190,203,204]。尽管该理论存在多个不同版本，最简单 (极小) 的理论其拉格朗日量为

$$\mathcal{L}_{(DF)^2+\mathrm{YM}} = \frac{1}{2}\left(D_\mu F^{a\mu\nu}\right)^2 - \frac{1}{4} m^2 \left(F^a_{\mu\nu}\right)^2, \tag{65}$$

where a is an adjoint index. There also exist variants of the theory with additional scalars in a specific matter representation [203-205].

其中 a 是一个伴随指标。该理论也存在在特定物质表示中加入额外标量的变体 [203-205]。

To give concrete examples, the right column of Table 4 gives a variety of double-copy theories where one copy is $\mathcal{N} = \mathcal{N}_1$ SYM theory with the second copy shown in the left column. These results are located on/close to the central hexagon of Fig. 6 with blue links representing SYM theory.

举具体例子，表 4 右栏给出了多种双拷贝理论，其中第一个拷贝是 $\mathcal{N} = \mathcal{N}_1$ SYM 理论，第二个拷贝在左栏给出。这些结果位于图 6 中心六边形及临近位置，蓝色连线代表 SYM 理论。

The first row corresponding to the previously explained double copy of two SYM theories leads to an ungauged supergravities with $N = N_1 + N_2$ supersymmetries. With $D = 4$, for $\mathcal{N} \leq 4$, the result is supergravities including matter multiplets; specifically, if $N_1 = N_2 = 2$, we obtain ungauged $\mathcal{N} = 4$ supergravity in four dimensions with two vector multiplets; if $N_1 = N_2 = 1$, we obtain $\mathcal{N} = 2$ supergravity in four dimensions with one hypermultiplet [4, 5, 11, 16, 105, 206] . A simple way to understand the appearance of different multiplets is by counting the number of resulting states in the double-copy tensor products of states.

第一行对应之前已经说明过的两个 SYM 理论的双拷贝，得到具有 $N = N_1 + N_2$ 个超对称的无规化超引力。当 $D = 4$ 时，对 $\mathcal{N} \leq 4$ ，结果是包含物质多重态的超引力；具体来说，若 $N_1 = N_2 = 2$ ，我们得到四维无规化 $\mathcal{N} = 4$ 超引力，带有两个矢量多重态；若 $N_1 = N_2 = 1$ ，我们得到四维 $\mathcal{N} = 2$ 超引力，带有一个超多重态 [4, 5, 11, 16, 105, 206] 。理解不同多重态出现的简单方法是计数双拷贝态张量积中得到的态数目。

The second row in Table 4 gives the double copy between the NLSM and SYM theory, which produces amplitudes of a supersymmetric version of the Dirac-Born-Infeld theory [53, 201]. This is not actually a gravitational theory: since the NLSM does not contain gluons, a graviton cannot be obtained from the double copy.

表 4 的第二行给出了非线性 sigma 模型与 SYM 理论之间的双拷贝，它生成超对称版本狄拉克-玻恩-因费尔德理论的振幅 [53, 201]。这实际上并不是引力理论: 由于非线性 sigma 模型不含胶子，无法通过双拷贝得到引力子。

Table 4 Multiplication table for purely adjoint double copies involving at least one SYM gauge theory with $\mathcal{N} = \mathcal{N}_1$ supersymmetry. The left column gives the second theory used in the double-copy construction. Theories are specified in D dimension. In case of the Yang-Mills-scalar theory from dimensional reduction, the theory is reduced from higher dimensions. See also Fig. 6

表 4 包含至少一个带有 $\mathcal{N} = \mathcal{N}_1$ 超对称的 SYM 规范理论的纯伴随双拷贝乘法表。左列给出了双拷贝构造中用到的第二个理论。理论在 D 维中定义。对于维度约化得到的杨-米尔斯-标量理论，该理论是从更高维度约化得到的。另见图 6。

QFT QFT	$\mathcal{N} = N_1$ SYM theory
$\mathcal{N} = N_2$ SYM	$\mathcal{N} = N_1 + N_2$ Maxwell-Einstein supergravity
NLSM	$\mathcal{N} = N_1$ Dirac-Born-Infeld theory
YM-scalar from dim. red.	$\mathcal{N} = N_1$ supergravity with matter vector multiplets
YM + ϕ^3 theory	$\mathcal{N} = N_1$ Yang-Mills-Einstein supergravity
$(DF)^2$ theory	$\mathcal{N} = N_1$ conformal supergravity

An important class of double-copy constructions arises when one of the gauge-theory factors is non-supersymmetric, as in the third row of Table 4. In this case, one has more freedom. The simplest option is the dimensional reduction of a pure Yang-Mills theory from $n + 4$ or $n + 5$ dimensions to four or five dimensions. The double copy with SYM theory leads in this case - according to the amount of supersymmetry - to either a $\mathcal{N} = 4$ ungauged supergravity with n vector multiplets or to a $\mathcal{N} = 2$ Maxwell-Einstein supergravity belonging to the so-called generic Jordan family [108] (see Refs. [152, 207] for a supergravity discussion of these theories).

当其中一个规范理论因子是非超对称的，就会得到一类重要的双拷贝构造，如表 4 第三行所示。在这种情况下，构造拥有更多自由度。最简单的选择是将纯杨-米尔斯理论从 $n+4$ 或 $n+5$ 维维度约化到四维或五维。根据超对称的数量，这种情况下与 SYM 理论的双拷贝会得到要么是带有 n 矢量多重态的 $\mathcal{N}=4$ 未规范超引力，要么是属于所谓一般若尔当族的 $\mathcal{N}=2$ 麦克斯韦-爱因斯坦超引力 [108](关于这些理论的超引力讨论见文献 [152, 207])。

An interesting variant to the latter construction is obtained by adding trilinear scalar coupling in Eq. (64) [108], as it appears on the fourth row of Table 4; the net result is either a $\mathcal{N}=2$ or a $\mathcal{N}=4$ Yang-Mills-Einstein supergravity in which the F^{IJK} -tensors from (64) are identified with the structure constants of the supergravity gauge group (see also Refs. [115, 208, 209]). Note that the R symmetry is untouched by this construction.

对上述后一种构造的一个有趣变体是在式 (64) 中加入三线性标量耦合 [108]，如表 4 第四行所示；最终结果是 $\mathcal{N}=2$ 或 $\mathcal{N}=4$ 杨-米尔斯-爱因斯坦超引力，其中来自式 (64) 的 F^{IJK} 张量被等同于超引力规范群的结构常数 (另见文献 [115, 208, 209])。注意 R 对称性不受该构造影响。

Finally, the minimal $(DF)^2$ theory enters the last row of Table 4. Together with SYM theory, its double-copy construction gives a mass deformation of conformal supergravity [203, 204, 210] ,

最后，最小 $(DF)^2$ 理论出现在表 4 的最后一行。它与 SYM 理论结合的双拷贝构造给出共形超引力 [203, 204, 210] 的质量形变,

$$\text{(mass-deformed minimal CSG)} = (\text{SYM}) \otimes \left(\text{minimal}\,(DF)^2 + \text{YM}\right). \tag{66}$$

The constructions outlined so far can be implemented at tree level, both by combining sets of duality-satisfying numerators as in Eq. 21 and by using the KLT formula 27.

到目前为止概述的构造都可以在树图阶实现，既可以像式 (21) 那样组合满足对偶性的分子集合，也可以使用 KLT 公式 (27)。

It is interesting to note that these types of double-copy constructions extend well outside the realm of ordinary supergravity theories. In particular, with a slightly different construction, we can obtain amplitudes from string theory. The starting point for this family of constructions is given by the set of disk integrals [211,212],

值得注意的是，这类双拷贝构造可以很好地拓展到普通超引力领域之外。特别地，只需稍微调整构造方式，我们就能得到弦论的振幅。这类构造的起点是一组圆盘积分 [211,212],

$$Z_\sigma\left(\rho\left(1,2,\ldots,n\right)\right) = (2\alpha')^{n-3} \int \frac{dz_1 \ldots dz_n}{\mathrm{vol}\left(\mathrm{SL}\left(2,\mathbb{R}\right)\right)} \frac{\prod_{i<j}^{n} \left|z_{ij}\right|^{\alpha' s_{ij}}}{\rho\left\{z_{12} z_{23} \cdots z_{n-1,n} z_{n,1}\right\}}.$$
$$\sigma\{-\infty \le z_1 \le z_2 \le \ldots \le z_n \le \ldots\},$$

(67)

In this equation $z_{ij} = z_i - z_j$ indicates the difference between the positions of the punctures. The vol(SL(2, $\mathbb{R}$) factor is handled by fixing the position of three punctures while introducing an appropriate Jacobian. The above integrals satisfy [212] the field-theory fundamental BCJ relations (19) with respect to the permutation ρ . Hence, they can be naturally contracted by the field-theory KLT kernel to give a well-defined double copy.

在该式中，$z_{ij} = z_i - z_j$ 表示刺点位置的差。vol(SL(2, $\mathbb{R}$)) 因子通过固定三个刺点的位置同时引入合适的雅可比行列式处理。上述积分满足 [212] 关于置换 ρ 的场论基本 BCJ 关系 (19)。因此，它们可以自然地用场论 KLT 核缩并，得到定义良好的双拷贝。

Taking SYM theory for the other set of partial ordered amplitudes, it can be shown that the output of the double-copy results in open-superstring amplitudes with color-ordered massless external states [211,212],

若取 SYM 理论作为另一组偏序振幅，可以证明，双拷贝的输出结果是带颜色序质量 less 外部态的开超弦振幅 [211,212],

$$A_{\mathrm{OS}}^{\text{tree}}\left(\sigma(1,2,\ldots,n)\right)=\sum_{\tau,\rho\in S_{n-3}(2,\ldots,n-2)} Z_{\sigma}(1,\tau,n,n-1)\, S[\tau\mid\rho]\, A_{\mathrm{SYM}}(1,\rho,n-1,n)\,. \tag{68}$$

The Z integrals have been interpreted as the amplitudes of a scalar theory dubbed Z-theory in Refs. [202,213,214]. A closed-string version of the Z-theory integrals is also available [215-218],

Z 积分在文献 [202,213,214] 中被解释为名为 Z 理论的标量理论的振幅。Z 理论积分也存在闭弦版本 [215-218],

$$\operatorname{sv} Z(\tau\mid\sigma)=\left(\frac{2\alpha'}{\pi}\right)^{n-3}\int\frac{d^2z_1\ldots d^2z_n}{\operatorname{vol}(\mathrm{SL}(2,\mathbb{C}))}\frac{\prod_{i<j}^{n}\left|z_{ij}\right|^{2\alpha' s_{ij}}}{\tau\left\{\bar z_{12}\cdots\bar z_{n-1,n}\bar z_{n,1}\right\}\sigma\left\{z_{12}\cdots z_{n-1,n}z_{n,1}\right\}}. \tag{69}$$

Here we consider ”sv” as part of the name of these integrals, but it can also be understood as an operation known as single-value projection (see Refs. [219, 220]). Using these building blocks, closed-superstring amplitudes can be constructed as double copies [221, 222]

我们这里将”sv” 视作这些积分名称的一部分，但它也可以理解为一种称为单值投影的操作 (见文献 [219, 220])。使用这些构造单元，闭超弦振幅可以构造为双拷贝 [221, 222]

$$(\text{closed superstring}) = (\mathrm{SYM})\otimes \operatorname{sv}(\text{ open superstring })\ . \tag{70}$$

This can also be understood as a ”triple copy”

这也可以被理解为” 三拷贝”

$$(\text{closed superstring}) = (\mathrm{SYM})\otimes \operatorname{sv}(\text{Z -theory})\otimes(\mathrm{SYM})\ . \tag{71}$$

Moreover, an analogous construction has been formulated for the open bosonic string [223]

此外，人们还为开玻色弦提出了一个类似的构造 [223]

$$
(\text{open bosonic string}) = \left((DF)^2 \text{ theory}\right) \otimes (\text{Z -theory}) \,. \tag{72}
$$

Relevant stringy double-copy constructions are summarized in Table 5.

相关的弦论双拷贝构造总结在表 5 中。

Many additional variants of the construction become available if one includes gauge theories with extra fields transforming in matter representations on top of the adjoint sector. In order to describe this family of constructions, one needs to spell out some additional rules on how fields in different representations enter the double-copy construction.

如果我们在伴随表示部分之外，引入包含携带物质表示额外场的规范理论，就可以得到该构造的诸多其他变体。为描述这一族构造，我们需要给出额外规则，说明不同表示的场如何进入双拷贝构造。

A common starting point is to consider gauge theories whose amplitudes can still be obtained in a presentation based on cubic graphs, i.e., there is no quartic or higher invariant symbol for the gauge group, so that only representation matrices, structure constants, and possibly Clebsch-Gordan coefficients with three indices from different representations appear in the color-dressed amplitudes. In this case, a color-kinematics-duality-satisfying presentation of amplitudes can still be obtained if numerator factors obey the same algebraic relations as the color factors, which now also include commutation relations for the representation matrices [6, 113].

通常的出发点是考虑仍可基于三次图写出振幅的规范理论: 也就是说，该规范群不存在四次或更高阶的不变符号，因此带色振幅中仅出现表示矩阵、结构常数，以及可能来自不同表示的三指标 Clebsch-Gordan 系数。在这种情况下，只要分子因子满足与颜色因子相同的代数关系 (此时代数关系也包含表示矩阵的对易关系 [6, 113])，我们仍能得到满足色运动学对偶的振幅表述。

Table 5 Multiplication table for stringy double copies involving at least one SYM [221-223]

表 5 至少包含一个 SYM 的弦论双拷贝乘法表 [221-223]

QFT String	SYM theory
Z-theory	Open superstring
sv(open superstring)	Closed superstring
sv(open bosonic string)	Heterotic string (gravity sector)

In order to combine fields in matter representations to construct double-copy states, we associate a gravitational state in the output of the construction to each gauge-invariant bilinear built out of gauge theory. In practice, this implies that no double-copy state is constructed as the product of an adjoint and a matter-representation state. The matter representations simply yield additional sectors in the supergravity theory. This criterion allows us to avoid the appearance of the extra spin-3/2 states that would otherwise lead to inconsistencies as discussed in section ”Color-Kinematics Duality and Supersymmetry.”

为了组合物质表示中的场来构造双拷贝态，我们将构造输出中的一个引力态对应到规范理论构造出的每个规范不变双线性项。实际上，这意味着不会用伴随态和物质表示态的乘积构造双拷贝态，物质表示只会为超引力理论带来额外的扇区。这一判据可以避免额外自旋 3/2 态的出现，正如“色运动学对偶与超对称性”一节讨论的，这类态会导致理论不自洽。

A concrete example of this family of double-copy constructions is given by the homogeneous $\mathcal{N} = 2$ supergravities in four and five dimensions. These theories were classified from the supergravity perspective by de Wit and van Proeyen in Ref. [224]. Their isometry algebras in four dimensions can be decomposed as $g = g_0 \oplus g_1 \oplus g_2$ defining

这类双拷贝构造的一个具体例子是四维和五维的齐次 $\mathcal{N} = 2$ 超引力。de Wit 和 van Proeyen 已经从超引力视角对这些理论做了分类，见文献 [224]。它们在四维的等度量代数可以分解为 $g = g_0 \oplus g_1 \oplus g_2$ ，由此定义

$$g_0 = so(1,1) \oplus so(q+2,2) \oplus S_q(P),$$

$$g_1 = (1,\ \text{spinor}, \text{vector}),$$

$$g_2 = (2,1,1). \tag{73}$$

Here $S_q(P)$ is a global symmetry group depending on the value of the parameter q and generators are labeled with their charge under $so(1,1)$. The corresponding isometry groups are not semisimple - they are instead given by the semidirect product between the subgroup generated by g_0 and the transformations corresponding to the other generators. Scalar manifolds for these theories are known as $L(q,P)$ spaces. In the particular case in which $q = 0(\bmod 4)$, an additional parameter $\dot{P}$ is needed to specify the theory. Among these homogeneous theories, some special cases give theories with symmetric scalar manifolds, which have additional symmetry generators and were previously classified in Refs. [152, 225, 226]. These are $P = 0, q$ generic and $q = 0, P$ generic, which both give the generic Jordan family, and $q = 1, 2, 4, 8$ with $P = 1$. The latter give the so-called magical supergravities. A double-copy construction for all homogeneous supergravities was formulated more recently in Ref. [51] and has the structure

这里 $S_q(P)$ 是依赖于参数 q 取值的整体对称群，生成元由其在 $so(1,1)$ 下的电荷标记。对应的等度量群不是半单群——它们由 g_0 生成的子群与其他生成元对应的变换构成半直积。这些理论的标量流形被称为 $L(q,P)$ 空间。当 $q = 0(\bmod 4)$ 时，需要额外的参数 $\dot{P}$ 来指定该理论。在这些齐次理论中，一些特殊情形给出了具有对称标量流形的理论，这类理论拥有额外的对称生成元，此前已在文献 [152, 225, 226] 中完成分类，分别是一般 $P = 0, q$ 和一般 $q = 0, P$ ，二者都给出一般若尔当族，以及带 $P = 1$ 的 $q = 1, 2, 4, 8$ ，后者给出所谓的神奇超引力。所有齐次超引力的双拷贝构造近期在文献 [51] 中给出，其结构为

$$\begin{pmatrix} N = 2 \text{ homogeneous} \\ \text{supergravity} \end{pmatrix} = \begin{pmatrix} N = 2 \text{ SYM} \\ + \frac{1}{2} \text{ hyper}_R \end{pmatrix} \otimes \begin{pmatrix} \text{YM from D dimensions} \\ + n_f \text{ fermions}_R \end{pmatrix}. \tag{74}$$

The supersymmetric gauge theory is $\mathcal{N} = 2$ SYM with a half-hypermultiplet transforming in a matter pseudo-real representation (Pseudo-reality of the representation is a necessary ingredient in the construction and ensures that the half-hypermultiplet can be introduced in the theory without having to be completed to a full hyper-multiplet; see Ref. [51].). The non-supersymmetric theory entering the construction is a Yang-Mills theory with n_f irreducible fermions in D dimensions reduced to four or five dimensions. The number of matter fermions and the dimension for the non-supersymmetric theory are related to the parameters given in the supergravity literature as $D = q + 6$ and $P = n_f$. Additionally, in some particular dimensions corresponding to $q = 0 (\mod 4)$, there exist two inequivalent spinor representations with different chirality; these are the cases in which the extra parameter $\dot{P}$ is needed. We refer the reader to Refs. [51, 227] for further details.

超对称规范理论是 $\mathcal{N} = 2$ SYM，带有一个处于物质伪实表示的半超多重态 (表示的伪实性是构造的必要条件，它保证半超多重态可以不用补全为完整超多重态就能引入理论中，见文献 [51])。参与构造的非超对称理论是杨-米尔斯理论，带有约化到四维或五维的 D 维 n_f 不可约费米子。物质费米子的数量和非超对称理论的维数与超引力文献中的参数满足关系 $D = q + 6$ 和 $P = n_f$ 。此外，在对应 $q = 0 (\mod 4)$ 的某些特定维数中，存在两个不等价、手性不同的旋量表示；正是这些情形需要额外参数 $\dot{P}$ 。进一步细节读者可以参考文献 [51, 227]。

Another important example is the double-copy construction for some gauged supergravities with Minkowski vacua [116, 119]. These theories are obtained as the double copy of a spontaneously broken gauge theory with an explicitly broken theory with massive fermions, after carefully tuning gauge-group representations and masses. While this family of constructions is still in its infancy, it provides an example of the subtleties one faces in extending the double copy to theories with several distinct matter representations.

另一个重要例子是部分具有闵氏真空的规范超引力的双拷贝构造 [116, 119]。在仔细校准规范群表示与质量后，这些理论可通过下述方式得到双拷贝: 一个自发破缺规范理论，与一个显式破缺且含大质量费米子的理论的双拷贝。尽管这类构造仍处于发展初期，它已经展示了将双拷贝推广到含多种不同物质表示的理论过程中会遇到的微妙问题。

The reader may wonder how is it possible to provide a double-copy construction for Einstein gravity. This requires removal of unwanted states produced by the double copy, which can be done by introducing matter fermions in one of the two Yang-Mills theories and matter ghost fields in the other [125, 228] or by explicit projection [229].

读者或许会好奇，如何为爱因斯坦引力构造双拷贝。这需要移除双拷贝产生的多余态，方法可以是在两个杨-米尔斯理论中的一个引入物质费米子，在另一个中引入物质鬼场 [125, 228]，也可以通过显投影实现 [229]。

Many additional examples of double-copy constructions have been formulated over the years beyond the ones mentioned in this section. These include theories with hypermultiplets [51, 52], Yang-Mills-Einstein theories with spontaneously broken gauge symmetry [113], various gravitational theories with higher-dimension operators [148, 149, 190, 230, 231], gravity with massive matter [120, 232, 233], massive (Kaluza-Klein) gravity [113, 234-236], heavy-mass effective theories [237, 238], and special constructions for supergravities in three dimensions [239- 241]. While here we have focused on BCJ double copies, similar results have been obtained with the so-called CHY formalism [201, 242, 243], as well as with ambitwistor strings [244-252] (see also the review [253]), the KLT bootstrap [254], formalisms based on differential operators [139, 255, 256], and approaches based on off-shell linearized (super)fields [257-262].

除本节提到的例子外，多年来人们还构造了许多其他双拷贝构造实例，包括含超多重态的理论 [51, 52]、规范对称性自发破缺的杨-米尔斯-爱因斯坦理论 [113]、各类含高维算符的引力理论 [148, 149, 190, 230, 231]、含大质量物质的引力 [120, 232, 233]、大质量 (卡鲁扎-克莱因) 引力 [113, 234-236]、重质量有效理论 [237, 238]，以及三维超引力的特殊构造 [239-241]。尽管本文聚焦于 BCJ 双拷贝，在所谓 CHY 形式体系 [201, 242, 243] 中也得到了类似结果，此外还有双矢引力子弦 [244-252](另见综述 [253])、KLTbootstrap[254]、基于微分算子的形式体系 [139, 255, 256]，以及基于脱壳线性化 (超) 场的方法 [257-262]。

Conclusions and Outlook

结论与展望

This chapter presented an overview of the on-shell perspective for supergravity theories [7-9], focusing on their ultraviolet properties and on the double-copy structure [3-6]. The double copy offers a radically different interpretation of perturbative gravity and relates supergravity and gauge-theory calculations at high-loop orders, providing practical means to determine explicitly the coefficient of potential ultraviolet counterterms.

本章概述了超引力理论的壳上视角 [7-9]，重点关注其紫外性质与双副本结构 [3-6]。双副本对微扰引力给出了完全不同的诠释，将超引力与高圈阶规范理论计算联系起来，为明确确定潜在紫外抵消项的系数提供了实用方法。

The question of ultraviolet divergences in supergravity theories has a long history. The dimensionful nature of Newton's constant suggests that all supergravity theories should diverge at some loop order. By their very nature, such power counting arguments can fail if symmetries or structures that induce nontrivial cancellations are not taken into account. This is not a hypothetical issue: at present there is no explanation for such enhanced cancellations. Examples include the absence of an ultraviolet divergence in $\mathcal{N} = 5$ supergravity at four loops [16], despite the existence of a counterterm that respects all symmetries manifestly preserved by regularized scattering amplitudes [22, 28, 195, 196]. Half-maximal supergravity at two loops in $D = 5$ is another such example [13,14,59,60]. An important characteristic of enhanced cancellations is that no local representation of the amplitude in terms of covariant diagrams can display them: the power counting of each diagram is necessarily worse than the power counting of the complete amplitude. In other words, the cancellations are highly nontrivial, and require an interplay between most, if not all, diagrams contributing to an amplitude, thus pointing to some novel structures or symmetries.

超引力理论中的紫外发散问题由来已久。牛顿常数具有量纲这一性质表明，所有超引力理论都应当会在某一圈阶发生发散。如果不考虑会诱导非平凡抵消的对称性或结构，这类幂计数论证本身就可能失效。这并非假想问题: 目前对这类增强抵消尚无解释。例子包括四圈阶 $\mathcal{N} = 5$ 超引力不存在紫外发散 [16]，尽管存在满足正则化散射振幅所有显式对称性的抵消项 [22, 28, 195, 196]。$D = 5$ 中两圈阶半最大超引力是另一个此类例子 [13,14,59,60]。增强抵消的一个重要特征是，协变图无法用任何振幅的局域表示展示这类抵消: 每个图的幂计数必然劣于完整振幅的幂计数。换言之，抵消是高度非平凡的，需要贡献振幅的绝大多数 (即便不是全部) 图之间相互作用，因此指向了某种全新结构或对称性。

The existence of enhanced cancellations suggests that further nontrivial surprises await us as we probe supergravity theories to ever higher-loop orders. A primary goal is to fully understand the origin of enhanced cancellations. For the case of half-maximal supergravity at two loops in $D = 5$, the double copy links the potential ultraviolet divergences to certain would-be divergences with forbidden color factors in pure Yang-Mills theory [13].

> 增强抵消的存在表明，当我们在越来越高的圈阶探究超引力理论时，还会有更多非平凡的惊喜等待我们。首要目标是完全理解增强抵消的起源。对于 $D = 5$ 中两圈阶半最大超引力而言，双副本将潜在紫外发散与纯杨-米尔斯理论中带有禁戒色因子的某类拟发散联系起来 [13]。

The enhanced cancellations that appear in $\mathcal{N} = 5$ supergravity at four loops in $D = 4$ is an important case because the amplitudes are free of the $U(1)$ anomalies that appear in $\mathcal{N} = 4$ supergravity [14, 61, 62] , which plausibly are an underlying source of divergences. An important challenge is to calculate the coefficient of the potential five-loop divergence in $\mathcal{N} = 5$ supergravity and ascertain whether enhanced cancellations continue beyond four loops. The generalized double copy [174, 178] provides a means to carry out this calculation. If this turns out to be finite, it should invigorate a search for an all-orders proofs of ultraviolet finiteness, analogous to the way calculations through three loops [34-37] stimulated proofs of finiteness for $\mathcal{N} = 4$ SYM theory [29-31]. On the other hand, if it turns out to be divergent at five loops, it would go a long way toward finally settling the question of whether ultraviolet-finite supergravity theories exist. Another important question relates to identifying a complete explanation for enhanced cancellations. Possible paths include an improved understanding of the consequences of supersym-metry, duality symmetries, or perhaps generalized symmetries [66-68]. Whatever explanation is ultimately found should be novel, given that the enhanced ultraviolet cancellations themselves are not of a type usually found in gauge theories.

> $D = 4$ 中四圈阶 $\mathcal{N} = 5$ 超引力出现的增强抵消是一个重要案例，因为该振幅不含 $\mathcal{N} = 4$ 超引力 [14, 61, 62] 中存在的 $U(1)$ 反常，而这类反常 plausibly 是发散的根本来源。一项重要挑战是计算 $\mathcal{N} = 5$ 超引力中潜在五圈发散的系数，确认增强抵消是否能延续到四圈以上。广义双副本 [174, 178] 为完成这一计算提供了方法。如果结果证明是有限的，这将推动人们搜索所有阶紫外有限性的证明，类似三圈计算 [34-37] 推动了 $\mathcal{N} = 4$ 超对称杨-米尔斯理论有限性证明 [29-31] 的方式。另一方面，如果结果证明五圈阶确实发散，那将大大推动最终解决紫外有限超引力理论是否存在的问题。另一个重要问题是找到增强抵消的完整解释。可行研究方向包括更深入理解超对称、对偶对称性或广义对称性的效应 [66-68]。考虑到增强紫外抵消本身就不属于规范理论中常见的类型，最终找到的任何解释都必然是全新的。

There are also challenges related to improving our understanding of the duality between color and kinematics and the associated double copy. Our ability to carry out high-loop computations relies on finding double-copy format of gravitational scattering amplitudes. In particular, no BCJ representation of the five-loop four-point $\mathcal{N} = 4$ SYM amplitude is known, making it necessary to use the more complicated generalized double-copy construction [17, 174, 178]. While this is sufficient for carrying out five-loop computations for $\mathcal{N} = 8$ supergravity [174], and likely for $\mathcal{N} = 5$ supergravity, such computations would be enormously simplified by having gauge-theory integrands that manifestly satisfy the duality between color and kinematics.

在深化我们对颜色-运动学对偶及其相关双副本的理解方面也存在挑战。我们开展高圈计算的能力依赖于找到引力散射振幅的双副本形式。具体而言，目前尚未找到五圈四点 $\mathcal{N} = 4$ 超对称杨-米尔斯振幅的 BCJ 表示，因此必须使用更复杂的广义双副本构造 [17, 174, 178]。尽管这足以开展 $\mathcal{N} = 8$ 超引力 [174] 的五圈计算，对 $\mathcal{N} = 5$ 超引力应当也足够，但如果拥有能显式满足颜色-运动学对偶的规范理论被积函数，这类计算的复杂度会大大降低。

One of the most interesting aspects of the double copy is that it gives nontrivial links between theories. This web of theories [6, 53, 133] links together both gravitational and nongravitational theories by sharing single-copy theories. The complete picture is still elusive and may ultimately answer the outstanding question of whether all supergravity theories can be expressed in a double-copy form [51,52].

双重拷贝最有趣的性质之一是它在不同理论之间建立了非平凡关联。这个理论网络 [6, 53, 133] 通过共享单拷贝理论，将引力理论与非引力理论连接在一起。完整图景目前仍不清晰，它最终或许能解答一个悬而未决的关键问题: 是否所有超引力理论都可以表示为双重拷贝的形式 [51,52]。

Another important outstanding question is to fully understand the underlying kinematic algebra [128, 136-147], which would be helpful for building multiloop supergravity integrands through the double copy. Further interesting related topics that we did not cover here include the construction of classical double-copy solutions, applications to gravitational-wave physics, and generalizations of color-kinematics duality and the double copy to massive theories and to theories with different representations of the gauge group. For further discussion we refer the reader to various reviews $[6, 135, 157, 263 - 267]$.

另一个重要的待解决问题是完全理解底层的运动学代数 [128, 136-147]，这将有助于通过双重拷贝构造多圈超引力被积函数。本文未涵盖的其他有趣相关课题包括: 经典双重拷贝解的构造、在引力波物理中的应用，以及色-运动学对偶与双重拷贝向有质量理论、带不同规范群表示的理论的推广。更多讨论读者可参考各类综述 $[6, 135, 157, 263 - 267]$ 。

In summary, while many interesting questions remain, the double-copy approach to (super)gravity has proven to be rather fruitful, such as finding nontrivial relations to gauge theories, providing a powerful route to high-loop calculations shedding light on the ultraviolet behavior of supergravity theories.

总而言之，尽管仍有诸多有趣问题待解答，但 (超) 引力的双重拷贝方法已被证明成果颇丰，例如它找到了双重拷贝与规范理论之间的非平凡关联，为高圈计算提供了强有力的途径，也帮助我们厘清了超引力理论的紫外行为。

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